Pattern formation in micromagnetics

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Magnetization

Model for mesoscopic magnetization $m$ of a ferromagnetic sample $\Omega$
Model

Energy:

\[ E = d^2 \int_\Omega |\nabla m|^2 \, dx \quad \text{exchange} \]

\[ + Q \int_\Omega m_2^2 + m_3^2 \, dx \quad \text{anisotropy} \]

\[ + \int_{\mathbb{R}^3} |H_{\text{str}}|^2 \, dx \quad \text{stray field} \]

\[ - 2 \int_\Omega H_{\text{ext}} \cdot m \, dx \quad \text{external field} \]

[next slides]
Non-convexity through saturation constraint:

\[ |m|^2 = 1 \quad \text{in} \quad \Omega \]

Non-locality through stray field:

\[
\begin{align*}
\n \nabla \cdot (m + H_{str}) &= 0 \\
\n \nabla \times H_{str} &= 0
\end{align*}
\] distributionally in \( \mathbb{R}^3 \)

\[
\int_{\mathbb{R}^3} |H_{str}|^2 \, dx = \int_{\mathbb{R}^3} \|\nabla\|^{-1} \nabla \cdot m|^2 \, dx
\]
Electrostatic analogy

\[ H_{\text{stray}} = -\nabla U \]

\[
\begin{cases}
-\nabla^2 U = -\nabla \cdot m & \text{in } \Omega \\
- \left[ \frac{\partial U}{\partial \nu} \right] = \nu \cdot m & \text{on } \partial \Omega \\
-\nabla^2 U = 0 & \text{in } \mathbb{R}^3 - \Omega
\end{cases}
\]

volume charges

surface charges

\[ H_{\text{stray}} \]

\[ -\nabla \cdot m \]

\[ \nu \cdot m \]
Investigated phenomena

- Domain branching
- Landau state
- Concertina pattern
- Cross-tie wall
Pattern 1: bulk sample, domain branching

\[ \Omega = \mathbb{R}^2 \times (0, t) \]

Bloch wall

Hubert Kohn & Müller
Choksi & Kohn, & O. Conti
Pattern 2: thin film, Concertina pattern

\[ \Omega = \mathbb{R} \times (0, l) \times (0, t) \]

Néel wall

Cantero & Steiner & O., & Schäfer & Wiezcorek
Pattern 3: thin film, Landau pattern

\[ \Omega = (0, l) \times (0, l) \times (0, t) \]

Bryant & Suhl
DeSimone &
Kohn &
Müller & O.,
& Schäfer,
& Drwenski
Pattern 4: thin film, cross–tie wall

\[ \Omega = \mathbb{R}^2 \times (0, t) \]

Bloch line = vortex

Alouges & Rivière & Serfaty, DKMO
Pattern 5: thin film, Néel walls

Symmetric vs. Asymmetric (LaBonte)

$m = m(x_1)$

$\{m_1(x_1)\}$

$m_1(x_1, 0)$

$h_{stray} = 0$

Optimal mix of symmetric tails and asymmetric core

as a function of the wall angle

Döring & Ignat & O.
Concertina pattern
Goal: birth and coarsening of Concertina
Sample geometry for Concertina

Thin film element: \( \Omega = \mathbb{R} \times (0, \ell) \times (0, t) \), period \( L \) in \( x_1 \)

external field: \( H_{\text{ext}} = (-h_{\text{ext}}, 0, 0) \)

stationary point: \( m = (1, 0, 0) \)
Experiments: period $w$ increases with width $\ell$

$\ell = 60\mu m$  

$\ell = 80\mu m$  

$\ell = 100\mu m$

thickness $t = 80nm$
Experiments: period $w$ decreases with thickness $t$

$t = 30\text{nm}$ \hspace{1cm} $t = 80\text{nm}$ \hspace{1cm} $t = 300\text{nm}$

width $\ell = 100\mu m$

... reproducible trends
Theory: Derivation and analysis of reduced model
Unstable mode and critical field $h_{crit}$

**Ansatz I:** coherent rotation

\[ h_{crit} \sim \frac{t}{\ell} \log(\ell/t) \]

**Ansatz II:** buckling

\[ h_{crit} \sim \left(\frac{d}{\ell}\right)^2 \]
Ansatz III: oscillatory buckling

\[ w \sim \left( \frac{d^2 \ell^2}{t} \right)^{1/3} \]

\[ h_{\text{crit}} \sim \left( \frac{d^2 t^2}{\ell^4} \right)^{1/3} \]

Is there a better Ansatz?

— rigorous answer by Ansatz–free lower bounds
There are exactly 4 regimes for $h_{\text{crit}}$ ...

**Theorem 1** (Cantero, O.). For $\ell \gg d$

\[
h_{\text{crit}} = f(L, T)
\]

\[
L := \frac{\ell}{d}
\]

\[
T := \frac{t}{d}.
\]

... Regime III is relevant for Concertina
Derivation of a reduced model ...

Limit in parameter space & blow–up in function space

... as singular limit
Rescaling for singular limit

Anisotropic rescaling of space:

\[
x_1 = \left( \frac{d^2 \ell^2}{t} \right)^{1/3} \hat{x}_1, \quad x_2 = \ell \hat{x}_2, \quad x_3 = t \hat{x}_3
\]

Blow up of magnetization:

\[
(m_2, m_3) = \left( \frac{d^2}{\ell t} \right)^{1/3} (\hat{m}_2, \hat{m}_3),
\]
Motivation of scaling in $m$ ...

\[ \nu' \cdot m' = 0 \]
\[ \nabla' \cdot m' = 0 \quad |m'|^2 = 1 \]
\[ \nu' \cdot m' = 0 \]

turns into

\[ \hat{m}_2 = 0 \]
\[ -\tilde{\partial}_1 \left( \frac{1}{2} \hat{m}_2^2 \right) + \tilde{\partial}_2 \hat{m}_2 = 0 \]
\[ \hat{m}_2 = 0 \]

... recovers Burgers operator
Reduced model …

**Theorem 2** (Cantero, O., Steiner).

\[
\int_0^1 \int_0^\hat{L} (\hat{\partial}_1 \hat{m}_2)^2 \, d\hat{x}_1 d\hat{x}_2
\]

exchange

\[
+ \frac{1}{2} \int_0^1 \int_0^\hat{L} \left( |\hat{\partial}_1|^{-1/2} \left( -\hat{\partial}_1 \left( \frac{1}{2} \hat{m}_2^2 \right) + \hat{\partial}_2 \hat{m}_2 \right) \right)^2 \, d\hat{x}_1 d\hat{x}_2
\]

stray field

\[
- \hat{h}_{ext} \int_0^1 \int_0^\hat{L} \hat{m}_2^2 \, d\hat{x}_1 d\hat{x}_2
\]

external field

… has single parameter \( \hat{h}_{ext} \)
Numerical simulation of reduced model
Simulation of reduced model: subcritical bifurcation

\[ \langle \hat{m}_2^2 \rangle^{1/2} \]

\[ \hat{h}_{crit} = 5.477 \]

unstable mode
Simulation of reduced model: turning point

\[ \langle \hat{m}_2^2 \rangle^{1/2} \]

\[ \hat{h}_{\text{ext}} = 6.8 \]
\[ \hat{h}_{\text{ext}} = 10.8 \]
\[ \hat{h}_{\text{ext}} = 30.8 \]

... birth of concertina
Simulation of reduced model: secondary bifurcations

\[ \hat{m}^2_{2} \] \[ ^{1/2} \]

\[ \hat{h}_{ext} = 6.09 \]
\[ \hat{h}_{ext} = 5.87 \]
\[ \hat{h}_{ext} = 6.96 \]

... coarsening of concertina
Comparison with experiment:
qualitative & quantitative
Material heterogeneity

Material: small grains, random easy axis

Reduced model: small, random external transversal field
Experiment vs. simulation of reduced model

Good qualitative agreement
Experiment vs. theory: Period of the concertina

average concertina period at formation

vs. period of unstable mode

ratio of
experimental
to theoretic

variations of theoretical \( w \)

Good agreement for large range of samples sizes

— systematic deviation of factor \( \varepsilon \in [1, 2) \) due to coarsening
Experiment vs. theory: difficult measurement

polycrystalline anisotropy

ripple

unstable mode

coarsened concertina

... capture right moment between ripple and coarsening
Experiment vs. theory: difficult measurement

Same sample at different $h_{ext}$ values

$t = 30\text{nm}$, $\ell = 40\text{nm}$

... capture right moment between ripple and coarsening
Theory: Coarsening
Eckhaus instability: the concept

Concavity of energy $\hat{E}$ per length in wave number $\hat{k} = \frac{2\pi}{\hat{w}}$

$\iff$ Concavity of energy $\hat{E}$ per period in period $\hat{w}$

$\iff$ Eckhaus instability
Eckhaus instability: valid despite non-locality

For fixed period \( \hat{w} \) consider

\[ \hat{m}_2 \text{ optimal configuration } \]
\[ \hat{E} \text{ minimal energy } \]
\[ \begin{aligned}
\text{among } \hat{w} \text{-periodic configurations }
\end{aligned}\]

Consider infinitesimal perturbations of Bloch form

\[ \delta m_2 = e^{i \hat{k}_1 \hat{x}_1} F(\hat{x}_1, \hat{x}_2) \text{ s.t. } F \text{ is } \hat{w} \text{-periodic in } \hat{x}_1 \]

**Theorem 3** (Steiner). Then for wave-length \( \hat{w} |\hat{k}_1| \ll 1 \)

\[ \inf_{\delta m_2} \frac{\text{Hess } \hat{E}(\hat{m}_2)(\delta \hat{m}_2, \delta \hat{m}_2)}{\int \int \delta \hat{m}_2^2 d\hat{x}_1 d\hat{x}_2} \approx \hat{k}_1^2 \frac{d^2}{d\hat{w}^2}(\hat{w} \hat{E}) \]
Bifurcation analysis: set-up

Small deviations from critical field and wave number

\[ \hat{h}_{ext} = \hat{h}^*_{ext} + \delta h_{ext} \quad \text{and} \quad \hat{k} = \hat{k}^* + \delta k \]

Perturbation of unstable mode

\[ \hat{m}_2 \approx A \hat{m}^*_2 + A^2 \hat{m}^{**}_2 \]

Optimization in \( \hat{m}^{**}_2 \) yields

\[ \hat{E} \approx -c_2 \delta h_{ext} A^2 - c_4 A^4 \]

Note \( c_4 = 0.0075 \ll 1 \)
Bifurcation analysis:
unfold near-degenerate bifurcation

Expanded Ansatz

\[ \tilde{m}_2 \approx A \tilde{m}_2^* + A^2 \tilde{m}_2^{**} + A^3 \tilde{m}_2^{***} \]

Optimization in \( \tilde{m}_2^{**} \) and \( \tilde{m}_2^{***} \) yields

\[ \hat{E} \approx (-c_2 \delta h_{ext} + \tilde{c}_2 \delta k^2) A^2 + (-c_4 + \tilde{c}_4 \delta k) A^4 + c_6 A^6 \]
Bifurcation analysis:
Optimal and Eckhaus stable periods

Given $\hat{h}_{ext} = \hat{h}_{ext}^* + \delta \hat{h}_{ext}$ determine $\hat{k} = \hat{k}^* + \delta \hat{k}$ s.t.

- Optimal: \[
\frac{\partial \hat{E}}{\partial A} = \frac{\partial \hat{E}}{\partial \delta k} = 0, \quad \frac{\partial^2 \hat{E}}{\partial A^2} > 0, \quad \text{det} \left( \begin{array}{cc} \frac{\partial^2 \hat{E}}{\partial A^2} & \frac{\partial^2 \hat{E}}{\partial A \partial \delta k} \\ \frac{\partial^2 \hat{E}}{\partial A \partial \delta k} & \frac{\partial^2 \hat{E}}{\partial \delta k^2} \end{array} \right) > 0
\]

- Marginally E.-stable: \[
\frac{\partial \hat{E}}{\partial A} = 0, \quad \frac{\partial^2 \hat{E}}{\partial A^2} > 0, \quad \text{det} \left( \begin{array}{cc} \frac{\partial^2 \hat{E}}{\partial A^2} & \frac{\partial^2 \hat{E}}{\partial A \partial \delta k} \\ \frac{\partial^2 \hat{E}}{\partial A \partial \delta k} & \frac{\partial^2 \hat{E}}{\partial \delta k^2} \end{array} \right) = 0
\]
Bifurcation analysis:
Optimal and Eckhaus stable periods

\[ \hat{w} = \frac{2\pi}{k} = \frac{2\pi}{k^* + \delta k} \]

\[ \hat{h}_{ext} = \hat{h}_{ext}^* + \hat{\delta h}_{ext} \]

All stable concertina patterns have period $> w^*$
Domain theory: Set-up

Solution of Burgers equation

\[ \hat{\sigma} = -\hat{\partial}_1 \frac{1}{2} \hat{m}_2^2 + \hat{\partial}_2 \hat{m}_2 = 0 \text{ distributionally,} \]

... selected by wall energy + external field:

\[ \hat{E}_{\text{Domain}} = \int_{\text{jump set}} e_{\text{wall}}([\hat{m}_2]) \, d\hat{x}_2 - \hat{h}_{\text{ext}} \int_0^1 \int_0^{\hat{L}} \hat{m}_2 \, d\hat{x}_1 \, d\hat{x}_2 \]
Domain theory: Néel wall energy

A two–scale object: core + extended tails

Asymptotic analysis + fitting to numerical simulation:

\[ e_{\text{wall}}(2\hat{m}_2^0) = \frac{\pi}{8} (\hat{m}_2^0)^4 \ln^{-1} \left( \frac{L}{c} (\hat{m}_2^0)^2 \right) \quad c \approx 1.8559 \]

Néel wall energy is quartic in wall angle
Domain theory: simple Ansatz

Ansatz functions of concertina type:

\[ \hat{E}_{\text{Domain}}(A, \hat{w}, \hat{h}_{\text{ext}}) \]

... yield explicit expression \( \hat{E}_{\text{Domain}}(A, \hat{w}, \hat{h}_{\text{ext}}) \)
Domain theory: scaling

\[
L^{-1} \hat{E}_{Domain}(A, \hat{w}, \hat{h}_{ext}) \approx (\hat{h}_{ext}^3 \ln^2 \hat{h}_{ext}) F \left( \frac{A}{\hat{h}_{ext} \ln \hat{h}_{ext}}, \frac{\hat{w}}{\hat{h}_{ext} \ln \hat{h}_{ext}} \right)
\]

... predicts coarsening, on level of minimizer and stability
Domain theory: scaling consistent with reduced model

**Theorem 4** (O., Steiner). $\hat{h}_{\text{ext}} \gg 1$ and $\hat{L} \gtrsim \hat{h}_{\text{ext}} \ln \hat{h}_{\text{ext}}$:

i) $\min \hat{L}^{-1} \hat{E} \sim -\hat{h}_{\text{ext}}^3 \ln^2 \hat{h}_{\text{ext}}$

ii) For any $\hat{m}_2$ with $\hat{L}^{-1} \hat{E}(\hat{m}_2) \sim \min \hat{L}^{-1} \hat{E}$ we have

\[
A^2 := \hat{L}^{-1} \int_0^1 \int_0^{\hat{L}} \hat{m}_2^2 \, d\hat{x}_1 \, d\hat{x}_2 \sim (\hat{h}_{\text{ext}} \ln \hat{h}_{\text{ext}})^2
\]

\[
\hat{L}^{-1} \int_0^1 \int_0^{\hat{L}} (\hat{m}_2(\cdot + \hat{w}, \cdot) - \hat{m}_2)^2 \, d\hat{x}_1 \, d\hat{x}_2 \lesssim \left( \frac{\hat{w}}{\hat{h}_{\text{ext}} \ln \hat{h}_{\text{ext}}} \right)^{1/2} A^2
\]
Both asymptotics...

Bifurcation analysis

... match and confirm coarsening by Eckhaus instability
Hysteresis loop of the concertina
Conclusions

The challenge: few mechanism (exchange, stray field) — wealth of pattern

Our approach: Identification of parameter regime, Derivation of reduced model, numerical simulation