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A regularity theory for the moving contact line

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Spreading droplet

driven by surface tension $\leftrightarrow$ limited by viscosity

quasi–static balance

vertical length $\ll$ horizontal length
Different types of viscous dissipation

\[ \partial_t h + \partial_x (h^n \partial_x^3 h) = 0 \]

fourth-order parabolic degenerate mobility

Hele-Shaw cell \hspace{2cm} Navier slip \hspace{2cm} No slip

\( n = 1 \) \hspace{2cm} \( n = 2 \) \hspace{2cm} \( n = 3 \)

good range \( n \in (0, 3) \) logarithmic contact line singularity
Different relations between surface tensions

Contact angle $\theta$ from Young’s law:

$$\sigma_{as} = \sigma_{al} \cos \theta + \sigma_{ls}$$

Two qualitatively different cases:

$$\theta = 0 \quad \text{or} \quad 0 < \theta \ll 1$$

$$\partial_x h = 0 \quad \text{at} \partial\{h > 0\} \quad \text{or} \quad (\partial_x h)^2 = 1 \quad \text{at} \partial\{h > 0\}$$
Different stationary and travelling wave solutions

\[ \partial_x h = 0 \]

\[ (\partial_x h)^2 = 1 \]

\[ \partial_x h = 0, \ n \in (0, \frac{3}{2}) \]

\[ \partial_x h = 0, \ n \in (\frac{3}{2}, 3) \]

\[ (\partial_x h)^2 = 1, \ n \in (0, 3) \]
Existence theory of weak solutions

... based on

Energy estimate \[ \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (\partial_x h)^2 \, dx = -\int_{\mathbb{R}} h^n (\partial_x^3 h)^2 \, dx \]

“Entropy” estimate \[ \frac{d}{dt} \int_{\mathbb{R}} \eta_n(h) \, dx = -\int_{\mathbb{R}} (\partial_x^2 h)^2 \, dx \]

(Bernis ’96, Grün ’01)

... restricted to zero contact angle besides (O. ’97)
Qualitative properties of weak solutions

Waiting time:
DalPasso & Giacomelli & Grün ('01),
Grün ('04), Giacomelli & Grün ('06)

Finite speed of propagation:
Bernis ('96), Hulshof & Shikov ('98),
Bertsch & DalPasso & Garcke & Grün ('98), Grün ('02)

Asymptotic behavior:
Giacomelli & O. ('02), Carrillo & Toscani ('02)

... but no uniqueness for instance
Here: the simplest model

\[ n = 1, \text{ i. e. } \partial_t h + \partial_x (h \partial_x^3 h) = 0 \text{ on } \{h > 0\} \]

\[ \theta = 0, \text{ i. e. } \partial_x h = 0 \text{ at } \partial \{h > 0\} \]
Our point of view: A free boundary problem

\[
\begin{align*}
    h &= 0 \\
    \partial_x h &= 0 \\
    \partial_x^3 h &= \frac{dX}{dt} \\
\end{align*}
\]

\[ h > 0 \]

\[ \partial_t h + \partial_x (h \partial_x^3 h) = 0 \]

fourth order parabolic + free boundary + 1 b. c.
Our goal: Strong solutions

Either short–time or close to stationary.
Here: close to stationary $h = \frac{1}{2}x^2$
An analogy: porous medium equation

\[
\begin{align*}
\frac{dY}{dt} &= -\partial_x h \\
\partial_t h - \partial_x (h \partial_x h) &= 0 \\
h > 0
\end{align*}
\]

second order parabolic b. c. + free boundary + 1 b. c.
Strong solutions for porous medium equation

Angenent ('88): 1–d, short-time existence, moving coordinate transform, weighted Hölder spaces, semi-group theory

Daskalopoulos, Hamilton ('98): 2–d, short-time existence, hodograph transform, weighted Hölder spaces, Safanov’s strategy

Koch ('01): multi–d, short–time existence + near self–similar solution, graph transform, weighted Hölder and \( L^p \)–spaces, singular kernel theory
Long–time existence close to stationary profile

$h_0$ closely in what sense?
A global transformation onto fixed domain

From \((t, x, X, h)\) to \((t, y, F)\):

\[ x = X(t) + y \quad \text{and} \quad h = \frac{1}{2}y^2 + yX + F \]

\[
\begin{align*}
\partial_x h &= y + X + \partial_y F \\
\partial^3 x h &= \partial^3 y F \\
\partial_t h + \partial_x h \frac{dX}{dt} &= \partial_t F + y \frac{dX}{dt}
\end{align*}
\]

\[
X = -\partial_y F|_{y=0} \\
\frac{dX}{dt} = \partial^3 y F|_{y=0}
\]

\[ 0 = \partial_t F - (\partial_y F - \partial_y F|_{y=0}) \partial^3 y F|_{y=0} \\
\quad + \partial_y \left((\frac{1}{2}y^2 + (F - y \partial_y F|_{y=0}))\partial^3 y F\right) \]
A simple form

All functions $F$ satisfy $F|_0 := F|_{y=0} = 0$

Algebraic structure:

$$\partial_t F + AF + B(F, F) = 0$$

with

$$AF = \partial_y \left( \frac{1}{2} y^2 \partial^3_y F \right),$$

$$B(G, F) = \partial_y \left( (G - y \partial_y G|_0) (\partial^3_y F - \partial^3_y F|_0) \right)$$
A nice linear operator, part I

Algebraic observation:  \( y^{m-1} \partial_y^m \partial_y (y^2 \partial_y^3 F) = \partial_y^2 (y^{m+1} \partial_y^2 \partial_y^m F) \)

... motivates weighted Sobolev spaces \( H_m \) for \( m \geq 1 \)

\[
\langle F, G \rangle_{H_m} := \int_0^\infty y^{m-1} \partial_y^m F \partial_y^m G \, dy \quad F|_0 = G|_0 = 0
\]

\( A \) is symmetric & positive w. r. t. \( \langle \cdot, \cdot \rangle_{H_1} \):

\[
\langle F, AF \rangle_{H_1} = \langle F, F \rangle_{H_3}.
\]

\( A \) is self–adjoint with \( \text{Domain}(A) = H_1 \cap H_3 \cap H_5 \)
A nice linear operator, part II

Algebraic observation: \( \partial_y(y^2 \partial_y^3 F) = y \partial_y^2(y \partial_y^2 F) \)

\( A^{1/2} \) is given by

\[
A^{1/2} F = -y \partial_y^2 F \quad \text{with} \quad \text{Domain}(A^{1/2}) = H_1 \cap H_3.
\]

Scale of norms:

\[
|F|_{H^m}^2 = \langle F, A^{(m-1)/2} F \rangle_{H_1}.
\]
A nice linear operator, part III

Solution of

\[ \partial_t F + AF = G \quad \text{and} \quad F(t = 0) = F_0 \]

satisfies for all \( m \geq 3 \)

\[ |F|_{L^\infty(H_m)} + |F|_{L^2(H_{m+2})} \lesssim |F_0|_{H_m} + |G|_{L^2(H_{m-2})} \]
Attempt to estimate quadratic operator, part I

Units:
\[ [\partial_t F] = [F][t]^{-1}, \quad [A F] = [F][y]^{-2}, \quad [B(F, F)] = [F]^2[y]^{-4} \]

Scale invariance: \( y = \lambda \hat{y}, \quad t = \lambda^2 \hat{t}, \quad F = \lambda^2 \hat{F} \)

Norm on initial data compatible with scaling: \( |F_0|_{H_4} \)

Is there an estimate
\[
|B(F, F)|_{L^2(H_2)} \lesssim \left( |F|_{L^\infty(H_4)} + |F|_{L^2(H_6)} \right)^2
\]
Attempt to estimate quadratic operator, part II

\[ |B(F, F)|_{L^2(H_2)} \lesssim \left( |F|_{L^\infty(H_4)} + |F|_{L^2(H_6)} \right)^2 \]

Because of \( B(F, F) = \partial_y \left( (F - y\partial_y F|_0) \left( \partial_y^3 F - \partial_y^3 F|_0 \right) \right) \), this requires at least

\[ |\partial_y^3 F|_{y=0}|_{L^2(dt)} \lesssim |F|_{L^\infty(H_4)} + |F|_{L^2(H_6)}, \]

\[ |\partial_y^3 F|_{y=0}|^2 \lesssim |F|^2_{H_6} = \int_0^\infty y^5 (\partial_y^6 F)^2 dy, \]

\[ |f|_{y=0}|^2 \lesssim \int_0^\infty y^5 (\partial_y^3 f)^2 dy. \]

**Counterexample:** \( f = \ln |\ln y| \) for \( y \ll 1 \).
Interpolation space: same scaling but stronger

\[ |F|_{H_m} = \left( \int_0^\infty \inf_{F = F_- + F_+} \left( s^{-2} |F_-|^{2}_{H_{m-1}} + s^2 |F_+|^{2}_{H_{m+1}} \right) \frac{ds}{s} \right)^{1/2} \]

\[ \lesssim \int_0^\infty \inf_{F = F_- + F_+} \left( s^{-2} |F_-|^{2}_{H_{m-1}} + s^2 |F_+|^{2}_{H_{m+1}} \right)^{1/2} \frac{ds}{s} \]

\[ =: |F|^{*}_{H_m} \gtrsim |\partial_y^{m/2} F|_{L^\infty} \]
Elementary characterization of $H^*_4$, I

Nonlinear change of variables

\[ F(y) \mapsto h(z) := \sqrt{2} z^{5/2} (\partial_y^3 F)(z^2) \]

\[ |F|_{H^3} \quad |h|_{L^2} \]
\[ |F|_{H^4} = \langle h, \frac{1}{4}(-\partial_z^2 + \frac{15}{4} \frac{1}{z^2})h \rangle_{L^2} \sim \langle h, (-\partial_z^2)h \rangle_{L^2} \]
\[ |F|_{H^5} = \langle h, \frac{1}{16}(\partial_z^4 - \frac{15}{2} \partial_z \frac{1}{z^2} \partial_z - \frac{135}{16} \frac{1}{z^4})h \rangle_{L^2} \sim \langle h, \partial_z^4 h \rangle_{L^2} \]

\[ \inf_{F=F_+ + F_-} (s^2 |F_-|_{H^3}^2 + s^{-2} |F_+|_{H^5}^2) \sim \langle h, s^2 (s^4 + \partial_z^4)^{-1} \partial_z^4 h \rangle_{L^2} \]
Elementary characterization of $H^*_4$, II

Extension

\[ h(z) \mapsto \bar{h}(z) := \begin{cases} h(z) & z \geq 0 \\ 0 & z \leq 0 \end{cases} \]

\[
\langle h, s^2 (s^4 + \partial^4_D)^{-1} \partial^4_z h \rangle_{L^2} \sim \langle \bar{h}, s^2 (s^4 + \partial^4_z)^{-1} \partial^4_z \bar{h} \rangle_{L^2}
\]

\[
= \langle \partial_z \bar{h}, s^2 \partial^2_z (s^4 + \partial^4_z)^{-1} \partial_z \bar{h} \rangle_{L^2}
\]
Elementary characterization of $H^*_4$, III

Decomposition in Fourier space

$$|F|_{H^*_4} \sim \int_0^\infty \frac{ds}{s} \left( \langle \partial_z \bar{h}, s^2 \partial_z^2 (s^4 + \partial_z^4)^{-1} \partial_z \bar{h} \rangle_{L^2} \right)^{1/2}$$

$$= \int_0^\infty \frac{ds}{s} \left( \int_{-\infty}^{\infty} \frac{s^2 |k|^2}{s^4 + |k|^4} |\mathcal{F}(\partial_z \bar{h})|^2 \, dk \right)^{1/2}$$

$$\sim \sum_{\ell \in \mathbb{Z}} \left( \int_{\{2^\ell \leq |k| < 2^{\ell+1}\}} |\mathcal{F}(\partial_z \bar{h})|^2 \, dk \right)^{1/2}$$

Compare to

$$|F|_{H^4} \sim |\partial_z h|_{L^2} \sim \left( \sum_{\ell \in \mathbb{Z}} \int_{\{2^\ell \leq |k| < 2^{\ell+1}\}} |\mathcal{F}(\partial_z \bar{h})|^2 \, dk \right)^{1/2}$$
Main result

Analogous definition

\[ |F'|_{L^p(H_m)^*} \begin{cases} |F|_{L^p(H_{m-1})} \\ |F|_{L^p(H_{m+1})} \end{cases} \]

**Theorem (GKO).** Provided \( |F_0|_{H^*_4} \ll 1 \), there exists a unique \( F \in L^\infty(H_4)^* \cap L^2(H_6)^* \) with

\[ \partial_t F + AF + B(F, F) = 0, \quad F(t = 0) = F_0. \]

It satisfies

\[ |F|_{L^\infty(H_4)^*} + |F|_{L^2(H_6)^*} \lesssim |F_0|_{H^*_4}. \]
Main estimate

**Proposition.** For all $F, G$ with $G|_0 = 0$

$$|B(G, F)|_{L^2(H_2)^*} \lesssim |G|_{L^\infty(H_4)^*} |F|_{L^2(H_6)^*}.$$ 

**Proof:**

Split $F$ into part $\tilde{F}$ with $\partial^3_y \tilde{F}|_0 = 0$ and $\partial^3_y F|_0(t) \xi(y)$.

Split $G$ into part $\tilde{G}$ with $\partial^2_y \tilde{G}|_0 = 0$ and $\partial^2_y G|_0(t) \eta(y).$
Split into two estimates

**Lemma 1.** For all $F, G$ with $G|_0 = \partial_y^3 F|_0 = 0$

\[ |B(G, F)|_{L^2(H_2)^*} \lesssim |G|_{L^\infty(H_4)^*} |F|_{L^2(H_6)^*}. \]

**Lemma 2.** For all $F, G$ with $G|_0 = \partial_y^2 G|_0 = 0$

\[ |B(G, F)|_{L^2(H_2)^*} \lesssim |G|_{L^\infty(H_4)^*} |y^4 \partial_y^8 F|_{L^2(L^1)}. \]
Proof of Lemma 1, part I

\[ |B(G, F)|_{L^2(H_2)}^* \lesssim |G|_{L^\infty(H_4)}^* |F|_{L^2(H_6)}^*. \]

Fix $G$, interpolate between

\[ |B(G, F)|_{L^2(H_1)} \lesssim |G|_{L^\infty(H_4)}^* |F|_{L^2(H_5)} \]
\[ |B(G, F)|_{L^2(H_3)} \lesssim |G|_{L^\infty(H_4)}^* |F|_{L^2(H_7)} \]

Disintegrate in time

\[ |B(G, F)|_{H_1} \lesssim |G|_{H_4^*}^* |F|_{H_5} \]
\[ |B(G, F)|_{H_3} \lesssim |G|_{H_4^*}^* |F|_{H_7} \]
Proof of Lemma 1, part II

\[ |B(G, F)|_{H_3} \]

\[ = \left| y \partial_y^4 \left( (G - y \partial_y G|_0) \partial_y^3 F \right) \right|_{L^2} \]

\[ \lesssim \left| y \partial_y^4 G \partial_y^3 F \right|_{L^2} + \cdots + \left| y (G - y \partial_y G|_0) \partial_y^7 F \right|_{L^2} \]

\[ \lesssim \underbrace{|y^{3/2} \partial_y^4 G|_{L^2}}_{\leq} \times \underbrace{|y^{-1/2} \partial_y^3 F|_{L^\infty}}_{\lesssim} + \cdots \]

\[ = |G|_{H_4} \lesssim |G|_{H^*_4} \lesssim |\partial_y^4 F|_{L^2} \lesssim |F|_{H_7} \]

\[ + \left| y^{-2} (G - y \partial_y G|_0) \right|_{L^\infty} \times \left| y^3 \partial_y^7 F \right|_{L^2} \]

\[ \lesssim |\partial_y^2 G|_{L^\infty} \lesssim |G|_{H^*_4} = |F|_{H_7} \]
Further results

Theorem’ (GKO). Provided $|F_0|_{H^*_4} \ll 1$,
the solution from Theorem 1 is in $C^\infty((0,\infty)_t \times [0,\infty)_x)$.

Use all $|\cdot|_{H_m}$. 
Progress by H. Knüpfer

\( n = 2, \ (\partial_x h)^2 = 1: \)

Asymptotics suggest \( h \approx x + c V x^2 \ln x \)

As before \( AF = \partial_y (y^2 \partial_y^3 F) \)

but now essential b. c. \( F|_0 = \partial_y F|_0 = 0 \)

\( n = 1, \ (\partial_x h)^2 = 1: \)

Rigorous lubrication approximation (Giacomelli & O. ’03)
Knüpfer & Masmoudi

Simplest open problem

\( n = 2, \ \partial_x h = 0: \)

Need to linearize around travelling wave \( h = (x + ct)^{3/2} \)