Shape Analysis on Lie Groups (and beyond) with Applications

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What is Shape analysis?

Many problems in object and activity recognition can be formulated in terms of similarities of *shapes*. By shapes we typically mean unparametrized curves in a vector space or on a manifold.

In recent years, a number of methods based on (infinite-dimensional) differential geometry have been developed to tackle problems such as

- Recognition of shapes (e.g. in image analysis),
- Interpolation and blending of shapes,
- Curve closing (e.g. in animation problems).

A popular approach is to define shapes as equivalence classes of certain mappings, where the equivalence relation is induced by reparametrization. Then one studies geometry on the "manifold of shapes" to develop efficient numerical algorithms for the above tasks. The challenge here is to construct geometries on shapes with good theoretical properties which still allow for efficient and accurate computations.

Why not use the L^2-distance?

The most natural distance on spaces of curves is perhaps the L^2-metric $\langle f, g\rangle_{L^2} = \int_I \langle \dot f(t), \dot g(t)\rangle \, \mathrm{d}t$. Unfortunately, the L^2-metric is very badly behaved on shape spaces. Its induced distance (measuring shortest curves) vanishes everywhere! Hence the L^2-metric does not lead to a usable distance.

Shapes, Metrics and the SRVT

Given curves $c_0, c_1 \colon I \to M$ with $I = [0,1]$ and M a vector space or manifold, define equivalence classes (also called *shapes*) $[c_0], [c_1]$ via:

$$c_0 \sim c_1 \iff \exists \varphi : c_0 = c_1 \circ \varphi,$$

where φ is a smooth, strictly increasing bijection on I. We denote by $\mathscr{P}$ the space of parametrized curves containing c_0 and c_1.

Typical choices of $\mathscr{P}$ include

- absolutely continuous functions,
- immersions,
- embeddings or
- piecewise linear functions.

The shapes, can then be collected in the corresponding *shape space*:

$$\mathscr{S} := \mathscr{P}/\sim.$$

Shape analysis concerns itself with the study of the spaces $\mathscr{S}$ and $\mathscr{P}$ from both a theoretical and a practical perspective.

We let $\mathscr{P}$ be the space of smooth immersions and define a *square root velocity transform* (SRVT)

$$\mathscr{R} \colon \mathscr{P} \to C^\infty(I, V\setminus\{0\}),\ \varphi \mapsto \frac{\alpha(\dot\varphi)}{\sqrt{\|\alpha(\dot\varphi)\|}}$$

where $\alpha\colon TM \to V$ is a smooth *transport map* (depending on M) into a vector space V.

Assuming that α is an immersion, we pull back the L^2-metric to obtain a Sobolev type metric on $\mathscr{P}$ which descents to a metric on $\mathscr{S}$. The induced distance

$$d_{\mathscr{P}}(c_0, c_1) = \sqrt{\int_I \|\mathscr{R}(c_0) - \mathscr{R}(c_1)\|^2 \, \mathrm{d}t}$$

measures bending and stretching of a curve while deforming it into another curve. This so called *elastic metric* leads to a rich and interesting geometry with "good" properties. Once the SRVT is understood, the distance is easy to compute.

To construct numerical methods and algorithms for shapes, results from infinite-dimensional differential geometry are needed. In a first step, one needs geometry on a manifold of shapes, i.e. a Riemannian metric. The metric can be obtained using a square root velocity transform (SRVT). This SRVT depends on a transport map and in general, there will be many different choices for such a map. If the target manifold carries additional geometric information (e.g. it is a vector space or a Lie group), then the geometry points to well behaved canonical choices for the transport. Our approach thus follows the slogan:

Geometry paves the way for numerical algorithms.

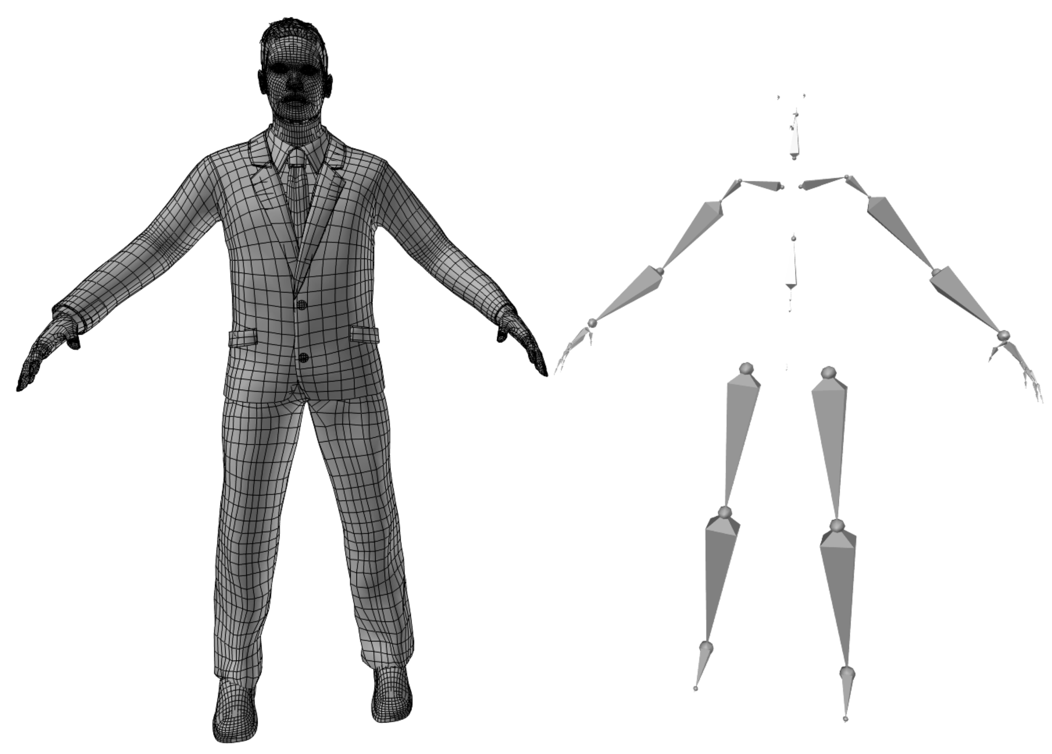
The Lie group case

Let $M = G$ be a Lie group. Then a natural choice for the transport is given by the group structure. Here we choose the vector space V to be the Lie algebra $\mathfrak{g}$ of G and define the transport map by right-translation of tangent vectors

$$\alpha\colon TG \to \mathfrak{g}, \quad v_g \mapsto v_g.g^{-1}$$

The corresponding SRVT is then a scaled version of the logarithmic derivative.

One possible application for our techniques comes from computer animation. Motions of virtual characters in movies and interactive applications are usually represented using a skeletal animation approach where the data consists of curves tracking the positions of the bones throughout the motion.



Figur: Surface mesh depicting a human character with underlying skeleton (only bones with degrees of freedom are shown)

Since the length of the bones in the skeleton is fixed, a motion of the skeleton can be described by the change of orientation of the bones in the skeleton to each other. If we disregard the position of the character in space, the motion is determined by a rotation of each joint (i.e. an element in the special orthogonal group $SO(3)$) for each time $t \in I$. Hence a motion corresponds to a smooth curve

$$\gamma\colon I \to SO(3)^{\#\text{joints}}.$$

which takes its image in a Lie group. Using methods for Lie group valued shape spaces, we can now compare and process motions.

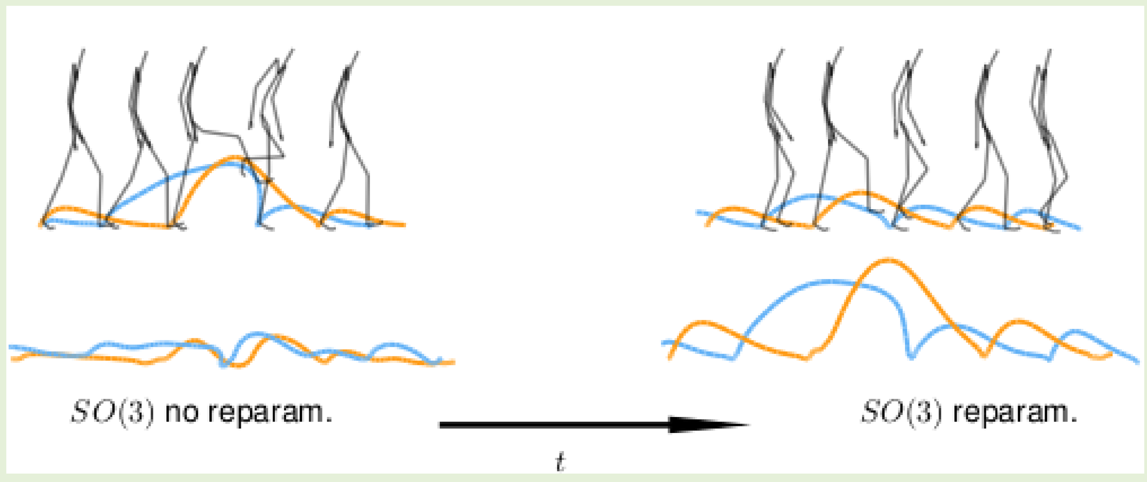
Limits of the motion model

The simple model of motions discussed above can be refined to include the position of the character in space without any conceptual difficulty. Note that the model does not force joints to bend realistically (i.e. not backwards). Realising these constraints is difficult.

Applications to computer animation

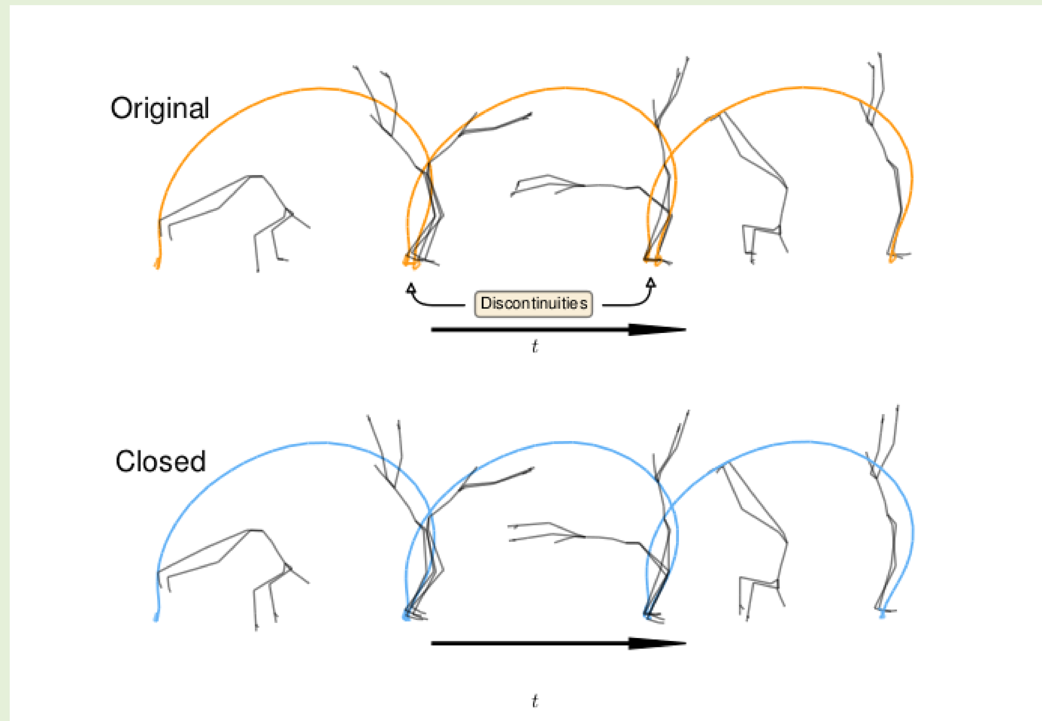
A typical approach of generating animations for virtual characters in videos or games is *motion capturing*, where an actor's motions are recorded and the underlying skeletal motion is extracted. In this way one obtains libraries of motion capturing data which have to be adapted to the specific uses. Two common problems of this approach are:

1. Pregenerated libraries will often not contain all desired motions. For example, if one desires a running motion of various speeds, one is stuck with the prerecorded speeds. This leads naturally to **motion blending**: A jogging animation can be seen as the interpolation between a running and a walking motion. Hence numerical methods for shape blending allow the generation of the desired motion. Below we record the result of the Lie group valued interpolation of stepping motions.



Figur: Interpolation between two stepping motions (upper row). Curves are trajectories of the feet. Shown are different results with/without reparametrizations (i.e. matching speed of curves).

2. Due to cost reasons recordings in motion capturing databases tend to be short. If a longer animation or an animation for an indeterminate amount of time (e.g., in a video game under a player's control) is desired, one needs to continuously loop the motion. This leads to the problem of (continuous) **curve closing**. Numerical methods for Lie group valued shape spaces have been developed to handle these problems. We demonstrate the results with the continuous closing of an animation.



Figur: Continuous closing of a handspring animation.

Perspectives for further research

There are two natural extensions of the techniques developed for the Lie group case:

1. Introduce constraints in the Lie group case (e.g. joints should not bend backwards).
2. Consider more general spaces with additional geometric information such as the so called homogeneous spaces (e.g. Spheres, Stiefel manifolds, hyperbolic spaces etc.).
   These spaces have important applications. For example, problems from micro-Doppler analysis of radar signals can be reformulated as shape analysis problems on a hyperbolic manifold.