What about statistical physics of asymptotical scale free distributions?

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In the era of networks and big-data it is now well recognized that complex or anomalous systems, dominated by long-range interactions and correlation persistent in time, are characterized by quasi-stationary distributions that differ from the well-known exponential family of Boltzmann-Gibbs and Gauss distributions. Typical situations are observed recurrently, for instance, in the study of small physical systems as well as in physical-like systems studied in the field of the biological, economic and social sciences.

To deal with these new phenomenologies, a possible approach, advanced by Tsallis [1] about three decades ago, consists to replacing the traditional Boltzmann-Gibbs-Shannon entropic form with its continuous deformation, controlled by one or more parameters, capable to capture the novelty observed in these anomalous systems, encoded in the distribution function of the meta-equilibrium without significantly modifying the epistemological structure of statistical mechanics [2].

Among the galore of the different entropic forms proposed in the existing literature, the Sharma-Taneja-Mittal entropy, originally advanced in the framework of the information theory, emerges as a family of generalized entropies modelled by two deformation parameters [3], which includes, as a subclass, the Tsallis entropy, the Kaniadakis entropy and others, which are used as paradigm to explore the consistence of the emerging theory.

The new formulation of statistical mechanics is based on two functionals: the STM-entropy

$$S_{\kappa,r}[p] = -\sum_i p_i \, \ln_{\kappa,r}(p_i) \ ,$$

that mimics the BGS-entropy by replacing the standard logarithm with its deformed version

$$\ln_{\kappa,r}(x) = \frac{x^{r+\kappa} - x^{r-\kappa}}{2\,\kappa} \ ,$$

with $\ln_{0,0}(x) = \ln(x)$, and a new quantity $\mathcal{J}_{\kappa,r}[p] = \mathcal{I}_{\kappa,r}[p] - 1$ where

$$\mathcal{I}_{\kappa,r}[p] = \sum_i p_i \, u_{\kappa,r}(p_i) \ ,$$

with

$$u_{\kappa,r}(x) = \frac{x^{r+\kappa} + x^{r-\kappa}}{2} \ ,$$

which vanishes in the $(\kappa,\, r) \to (0,\, 0)$ limit and therefore is irrelevant in the canonical theory. Functional $\mathcal{J}_{\kappa,r}[p]$ satisfies most of the axiomatic properties that an entropic form should preserved even if it is not properly entropy, but rather can be expressed as the difference between the STM-entropy and the so called para-entropy, a scaled version of STM-entropy, and is related to its Legendre transform in the probability space. The functional, $\mathcal{I}_{\kappa,r}[p]$ (or $\mathcal{J}_{\kappa,r}[p]$) plays a fundamental role in the mathematical formulation of the new theory since it is related to the definition of the thermodynamics potentials like the free energy or the Massieu potential, as well as to the corresponding partition function and allows a consistence formulation of the Legendere structure of the thermodynamics theory.

In this talk, I present the main aspects of statistical physics based on the STM-entropy, running from its derivation on the physical ground [3] to the associated algebraic structures working in the probability space and in the target space [4], its applications to the microcanonical and canonical assemblers [5] and its approach to the equilibrium by means of (non-linera) Boltzmann and Fokker-Planck equations, up to its recent reformulation in the framework of information geometry [5] in which the generalized exponential family, with the appropriate embedding, defines a dually flat statistical manifold with an Hessian structure that is consistent with the Legendre structure of the thermodynamics potentials obtained in the thermostatistics framework.

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