On the canonical distributions of the thermal particles in a weakly confining potential

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Abstract

Information geometry (IG) [1] provides us a powerful framework for studying a family of probability distributions by identifying the space of probability distributions with a differentiable manifold endowed with a Riemannian metric and an affine connection which is not Levi-Civita connection. An exponential family of probability distributions is most familiar in IG, and it played an important role especially in early developments of IG. In recent years much attention has been paid for studying the information geometric structured of some deformed exponential families of probability distributions. Some of them are Tsallis' q-deformed exponential [2] and Kaniadakis' κ-deformed exponential families [3] in non-extensive statistical mechanics. The κ-deformed exponential function is defined by

$$\exp_\kappa(x) := \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}, \tag{1}$$

for a real deformed parameter κ. The κ-deformed exponential function and its inverse function, i.e., κ-deformed logarithmic function, are important ingredients of the generalized statistical physics based on κ-entropy [3]. In addition some operators are also deformed by using these κ-deformed functions. For example, the κ-deformed sum is defined by

$$x \overset{\kappa}{\oplus} y := x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2}, \tag{2}$$

which reduces to the standard sum $x+y$ in the limit of $\kappa \to 0$.

Naudts' ϕ-exponential function [4] is a unified deformed-exponential function which includes the q- and κ-exponential function as special cases. We had studied some IG structures on the κ-deformed exponential families of probability distributions, which are non-Gaussians and with heavy-tails. For the κ-deformed exponential families, we constructed the suitable statistical manifolds and showed some information geometric structures such as κ-generalized Fisher metrics, θ- and η-potentials, dually-flat structures, κ-generalized divergence functions, and so on [5, 6, 7].

On the other hand, Fokker-Plank equation (FPE) is one of the most fundamental equations in statistical physics, and it is well known that for a thermal particle which is diffusing in a harmonic potential, the steady state solution of the corresponding linear FPE is Gaussian distribution. Harmonic (or parabolic) potential is a strongly confining potential, from which any particle never escape. In stead of such a strongly confining potential, we consider a thermally diffusing particle in a weakly confining potential defined by

$$V(x;x_c) := \frac{x_c^2}{2}\,\text{arsinh}\left(\frac{x^2}{x_c^2}\right), \tag{3}$$

with a control parameter x_c. In the limit of $x_c \to \infty$, this potential reduces to a parabolic potential $x^2/2$. Accordingly $V(x; x_c)$ can be considered as a deformation of the parabolic potential $x^2/2$. For a finite x_c this potential is weak (or shallow) in order to confine a thermal particle. Consequently when a particle has an enough amount of thermal energy, it can escape from a weakly confining potential. The corresponding FPE is still linear but has a nonlinear drift force caused by this nonlinear potential $V(x; x_c)$ [8].

In this contribution we consider the thermal probability distributions for the weakly confining potential $V(x; x_c)$ of Eq. (3) in the basic framework of statistical physics. In contrast to the well-known standard case of Gaussian distribution for strongly confining potential $x^2/2$, it is found that the quasi-equilibrium thermal probability distribution for this weakly confining potential $V(x; x_c)$ is non-Gaussian with heavy-tails. The corresponding FPE of the thermal probability distribution for this weakly confining potential describes an anomalous diffusion (or anomalous transport) in a parameter region in which the second moment $\langle x^2 \rangle$ diverges [8].

In addition we relate the canonical distribution for this weakly confining potential to the κ-exponential distribution

$$p(x) \propto \exp_\kappa \left(-\beta \frac{x^2}{2} \right), \tag{4}$$

for the strongly conning potential $x^2/2$ by introducing a suitable re-parameterization of the control parameter x_c. We further discuss some relations with the associated κ-deformed sum (2) and information geometric structures.

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