Some Information Inequalities for Statistical Inference

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In the context of nonextensive statistical mechanics, Naudts (2004) introduced a generalized notion of Fisher information and thereby a generalized Cramer-Rao lower bound. This is done by considering a pair of families of probability density functions (pdf), the original model and the *escort* model. In this article, we explore the statistical implications of this generalization. Further we state a generalized notion of the well known information inequalities in parameter estimation such as the Hammersley-Chapman-Robbins bound and Bhattacharyya Bounds in both regular and non-regular cases.

First we rewrite Naudt's generalized Cramer-Rao lower bound as follows. Let X be a random variable distributed according to some pdf $f(x, \theta)$, where $\theta \in \Theta \subseteq \mathbb{R}$. For convenience we restrict ourselves to the case of 1-dimensional parameter space. Let $g(x, \theta)$ be any density function which is also parametrized by the parameter $\theta \in \Theta$. Suppose the range of X is the set $A \subseteq \mathbb{R}$. Let us denote the expectation with respect to f_θ and g_θ by E_{f_θ} and E_{g_θ} respectively. Define two classes of estimators as

$$\mathcal{U}_f = \{U(X)/E_{f_\theta}(U) = 0\,;E_{f_\theta}(U^2) < \infty\ \forall\ \theta \in \Theta\} \tag{1}$$

$$\mathcal{U}_g = \{U(X)/E_{g_\theta}(U) = 0\,;E_{g_\theta}(U^2) < \infty\ \forall\ \theta \in \Theta\} \tag{2}$$

Assume that the probability measure P_{g_θ} is absolutely continuous with respect to the probability measure P_{f_θ} and $\mathcal{U}_f \subseteq \mathcal{U}_g$. To estimate a scalar function ϕ of θ, let $\mathcal{C}_\phi = \{S(X)/E_{f_\theta}(S(X)) = \phi(\theta), \forall\ \theta \in \Theta\}$. Also assume that $g(x, \theta)$ satisfies the regularity conditions, $g'(x, \theta) := \frac{dg(x,\theta)}{d\theta}$ exists for all $\theta \in \Theta$ and differentiation under integral sign is permitted. Naudts(2004) defined a generalized Fisher information as

$$N(\theta) = \int_A \frac{(g'(x, \theta))^2}{f(x, \theta)} dx \tag{3}$$

Theorem 1 *Suppose $N(\theta) < \infty$. Then for all $T(X) \in \mathcal{C}_\phi$*

$$\mathrm{Var}_{f_\theta}(T(X)) \geq \frac{(\lambda'(\theta))^2}{N(\theta)}, \quad \text{where } \ E_{g_\theta}[T] = \lambda(\theta). \tag{4}$$

We have many interesting examples in which the above bound is optimal. We describe two of them here.

Example 1 *Suppose $Y_1, \cdots, Y_n$ are i.i.d uniform random variables in $[0, \theta]$, where $\theta > 0$. Then*

$$X = \max\{Y_1, \cdots, Y_n\} \sim f(x, \theta) = \frac{nx^{n-1}}{\theta^{n-1}} \tag{5}$$

Then the unbiased estimator $T(X) = \frac{(n+1)X}{n}$ of θ attains the bound in (4) if we choose $g(x,\theta) = \frac{n(n+1)(1-\frac{x}{\theta})x^{n-1}}{\theta^n}$. When $n = 1$, this example reduces to Example 1 given in Naudts (2004).

Example 2 *Consider Gamma distribution $f(x, \theta) = \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1}e^{-x/\theta}}{\theta^\alpha}$, $\alpha > 0, \theta > 0$. Let $T(X) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)} X^k$, where k is a positive integer. Then T is an unbiased estimator θ^k and it attains the bound in (4) if we choose*

$$g(x, \theta) = \frac{1}{c} \frac{e^{-x/\theta}}{\theta} \left[\sum_{i=0}^{k-1} s_i \, (\frac{x}{\theta})^{\alpha+k-(i+2)} \right], \quad c = \sum_{i=0}^{k-1} s_i \Gamma(\alpha + k - (i+1)) \tag{6}$$

where $s_i = \prod_{j=1}^{i} (\alpha + k - j); i = 1, \cdots, k-1$ and $s_0 = 1$.
Also $T = 1/X$ attains the Naudt's lower bound for $g(x, \theta) = \frac{1}{\Gamma(\alpha-1)} \frac{x^{\alpha-2}e^{-x/\theta}}{\theta^{\alpha-1}}$. Note that $f(x, \theta)$ is an exponential family and $T = 1/X$ does not attain the Battacharya bounds of any order.

In addition, $\forall$ integers $k \geq 1$, we obtained a general information inequality based on the divided difference of $g(x, \theta)$, applicable even when g is not regular. When $f = g$, this bound reduces to the Bhattacharya bound in non regular case given by Fraser and Guttman (1952). When g is regular and $k = 1$, the bound reduces to Naudt's generalized Cramer-Rao lower bound.

References

[1] Fraser, D. A. S. and Guttman, I. (1952). Bhattacharyya Bounds without Regularity Assumptions. *Annals of Mathematical Statistics* , Vol 23(4), 629-632.

[2] Naudts, J. (2004). Estimators, escort probabilities, and ϕ-exponential families in statistical physics. *Journal of Inequalities in Pure and Applied Mathematics*, 5(4), Article 102.