Entropies associated with attractors at transitions to chaos in low-dimensional nonlinear dynamics

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We demonstrate that dual entropy expressions of the Tsallis type apply naturally to statistical–mechanical systems that experience an exceptional contraction of their configuration space. The entropic index $\alpha > 1$ describes the contraction process, while the dual index $\alpha' = 2 - \alpha < 1$ defines the contraction dimension at which extensivity is restored. We study this circumstance along the three routes to chaos in low-dimensional nonlinear maps where the attractors at the transitions, between regular and chaotic behavior, drive phase-space contraction for ensembles of trajectories. We illustrate this circumstance for properties of systems that find descriptions in terms of nonlinear maps.

It is generally acknowledged that the validity of ordinary, Boltzmann–Gibbs (BG), equilibrium statistical mechanics rests on the capability of a system composed of many degrees of freedom to transit amongst its many possible configurations in a representative manner. The number of configurations of a typical statistical–mechanical system increases exponentially with its size, and when these configurations are reachable in an adequate fashion through a sufficiently long time period, the indispensable BG properties, ergodicity and mixing, are established [1]. Therefore, to explore the limit of validity of BG statistical mechanics it is relevant to look at situations where access to configuration space can be controlled to various degrees down to a residual set of vanishing measure. A classic example is that of supercooled molecular liquids where glass formation signals ergodicity breakdown [2].

We refer to an especially tractable family of model systems in which the effect of phase space contraction in their statistical–mechanical properties can be studied theoretically. These are one-dimensional nonlinear maps that describe the three different routes to chaos, intermittency, period doublings and quasi periodicity [3]. Because these systems are dissipative they possess families of attractors, and the dynamics of ensembles of trajectories towards these attractors constitute realizations of phase space contraction. When the attractors are chaotic the contraction reaches a limit in which the contracted space has the same dimension as the initial space, a set of real numbers. But when the attractor is periodic the contraction is extreme and the final number of accessible configurations is finite. When the attractors at the transitions to chaos are multifractal sets contraction leads to more involved intermediate cases. Chaotic attractors have ergodic and mixing properties but those at the transitions to chaos do not [4]. We consider them here to discuss their association with generalized entropies.

The dynamical properties imposed by the attractors at the mentioned transitions to chaos in low-dimensional nonlinear maps can be easily determined [5] and it is our purpose to describe these properties in terms of phase space contraction. The attractors at the transitions to chaos provide a natural mechanism by means of which ensembles of trajectories are forced out of almost all phase space positions and become confined into a finite or (multi)fractal set of permissible positions. Such drastic contraction of phase space leads to nonergodic and nonmixing dynamics that is described by the dual entropic indexes $\alpha > 1$ and $\alpha' = 2 - \alpha < 1$. The first fixes the deformation of the exponential that measures the degree of contraction along (iteration) time evolution, and the second defines a contraction dimension, such that extensivity of entropy is restored. The dual entropy expressions are compatible with the same maximum entropy principle. When the contraction of phase space leads to a set of configurations of the same measure as the original phase space, e.g., an interval or finite collection of intervals of real numbers, one has $\alpha = \alpha' = 1$, the entropy expressions are the same and maintain the usual BG expression.

We chose to examine the statistical–mechanical effect of configuration space contraction at the renowned transitions to chaos in low-dimensional nonlinear maps, as these are perhaps the simplest situations where ergodicity and mixing properties breakdown. But in their own, the properties of these model systems manifest in natural phenomena. There are abundant examples of ranked data that obey (approximately) the empirical Zipf power law and these have been shown to comply with the tangent bifurcation property [6].

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