Deformed exponential families in statistical physics and beyond

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These days, traditional concepts from statistical physics and thermodynamics have a considerable impact on the development of the rather new subdiscipline of mathematical statistics, called information geometry. At the same time, the interests of statistical physicists broaden in various directions some of them gathered under the umbrella of complex systems. The interactions with other disciplines lead to a deepening of our understanding of the foundations of statistical physics.

The corner stone of equilibrium statistical physics is the Boltzmann-Gibbs distribution. It involves a Hamiltonian H, which fixes the physical model one is interested in, and a small number of parameters such as the inverse temperature β and the chemical potential μ. In the eyes of a mathematician the Boltzmann-Gibbs distribution is nothing but a parametrized probability distribution and, as a model, it belongs to the so-called exponential family of statistical models. In the context of information geometry the assumption is made that the probability distribution is a point x of a differentiable manifold, all points of which are reached by variation of the parameters.

The geometry of the statistical model is described by a metric tensor $g(x)$ together with a notion of geodesics. Both notions are well-known in physics. In classical mechanics space is flat. The metric tensor is trivial and everywhere the same. The path followed by the system is a geodesic. It is calculated by solving the Euler-Lagrange equations. The coefficients ω^k_{ij} appearing in these equations determine what mathematicians call the connection of the manifold. If taken equal to the Christoffel symbols $\omega^k_{ij} = \Gamma^k_{ij}$ then the connection is that of Levi-Civita. From the connection coefficients one can calculate the curvature of the manifold.

An appropriate choice of metric tensor is the Fisher-Rao metric. Fisher information plays an important role in statistics but is less known in the statistical physics community, where entropy is the starting concept. Other thermodynamic potentials are introduced depending on the context. Examples are the free energy F as a function of temperature T or the logarithm of the partition function Z as a function of β and further parameters such as the chemical potential μ. The matrix of second derivatives of the latter is the metric tensor.

The application of differential geometry to thermodynamics was initiated by Weinhold in 1975 and Ruppeiner in 1979. Points of the manifold are thermodynamic states. Ruppeiner discovered a relation between the curvature of this manifold and correlations between variables in the microscopic description of the physical model. In particular, in certain models the Gaussian curvature is found to diverge at the second order phase transition. This geometric insight might lead to a better understanding of phase transitions.

About 30 years ago Amari [1, 3] discovered that a systematic modification of the Levi-Civita connection of a statistical manifold removes the curvature. This leads to the theory of the dually flat geometries, one of which is the trivial geometry. An example is worked out during the talk.

A rather superficial analogy of one aspect of Amari's geometry with Tsallis' theory of q-deformed Gibbs ensembles [2] was for me the incentive to study the geometry of deformed exponential families [4, 6]. The q-deformed logarithm and exponential, introduced by Tsallis, can be further generalized and the resulting theory has been unified with the rho-tau formalism of Zhang [5].

The Boltzmann-Gibbs distribution is very successful in equilibrium statistical physics. However, in systems far from equilibrium the focus shifts toward dynamical equations such as those of Boltzmann, Fokker-Planck, Langevin. The application of methods of information geometry to the manifolds on which solutions of these equations propagate is the topic of ongoing research. Deformed exponential families can be used to model such manifolds.

References

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