On thermodynamic efficiency of collective computation

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Self-organisation of coherent motion in systems of self-propelled particles (e.g., flocks, swarms, active matter) is a pervasive phenomenon observed in many biological, chemical and physical settings [1, 2, 3, 4]. The ubiquity of collective motion across a range of different systems may be explained by some underlying universal principles. We interpret self-organisation of collective motion as an example of collective and distributed computation, and study it as a thermodynamic phenomenon, in the context of the first law of thermodynamics [5]. Recently, Bialek et al. [6, 7] provided a statistical mechanical model for the propagation of directional order throughout flocks. Despite these fundamental contributions, key thermodynamic quantities such as free entropy and work, dynamics of which are of special interest during critical regimes, are not explicitly incorporated in current statistical mechanical approaches to collective motion. In this abstract we report on our investigation of these quantities, exemplified using the dynamical model of collective motion proposed by Grégoire and Chaté [8]. This model exhibits a kinetic phase transition over the parameters controlling the particles' alignment, separating (i) the "disordered motion" phase, in which particles do not settle on a dominant direction while sharing a fairly regular collective space, and (ii) the "coherent motion" phase, in which particles cohesively move in a common direction. We analyse the dynamics of fundamental thermodynamical quantities, such as the generalised work, heat and energy, over a quasi-static process.

The Fisher information [9] measures the amount of information that an observable random variable $\mathcal{X}$ carries about unknown parameters $\theta = [\theta_1, \theta_2, \ldots, \theta_M]$. The probability of the states of the system, described by the state functions $X_m(x)$ over the configuration space and thermodynamic variables θ_m, in a stationary state, is given by the Gibbs measure:

$$p(x|\theta) = \frac{1}{Z(\theta)} e^{-\beta H(x,\theta)} = \frac{1}{Z(\theta)} e^{-\sum_m \theta_m X_m(x)}, \tag{1}$$

where $\beta = 1/k_b T$ is the inverse temperature T (k_b is the Boltzmann constant), the Hamiltonian $H(x,\theta)$ defines the total energy at state x, and $Z(\theta)$ is the partition function [10, 11]. The Gibbs free energy of such system is:

$$G(T,\theta_m) = U(S,\phi_m) - TS - \phi_m \theta_m, \tag{2}$$

where U is the internal energy of the system, S is the configuration entropy and ϕ_m is an order parameter. For a physical system described by the Gibbs measure in Eq. (1), the Fisher information has several physical interpretations, e.g., it is equivalent to the thermodynamic metric tensor $g_{mn}(\theta)$, measures the size of the fluctuations about equilibrium in the collective variables X_m and X_n, is proportional to the curvature of the free entropy $\psi = \ln Z = -\beta G$, and to the derivatives of the corresponding order parameters with respect to the collective variables [10, 12, 13, 11, 14]:

$$F_{mn}(\theta) = g_{mn}(\theta) = \Big\langle (X_m(x) - \langle X_m \rangle)(X_n(x) - \langle X_n \rangle) \Big\rangle = \frac{\partial^2 \psi}{\partial \theta_m \partial \theta_n} = \beta \frac{\partial \phi_m}{\partial \theta_n}, \tag{3}$$

where the angle brackets represent average values over the ensemble. Information-geometrically, the Fisher information is a Riemannian metric for the manifold of thermodynamic states, providing a measure of distance between thermodynamic states. It has also been argued that the difference between curvatures of the configuration entropy and the free entropy is related to a computational balance between uncertainty and sensitivity [15]. We establish a thermodynamic basis for this relationship as follows [5]:

$$\frac{d^2 \langle \beta U_{gen} \rangle}{d\theta^2} = \frac{d^2 S}{d\theta^2} - \frac{\partial^2 \psi}{\partial \theta_m \partial \theta_n} = \frac{d^2 S}{d\theta^2} - F(\theta), \tag{4}$$

where $\langle U_{gen} \rangle = U(S,\phi) - \phi\theta$. Under a quasi-static protocol, the first law of thermodynamics yields another important result for the generalised work W_{gen}:

$$F(\theta) = -\frac{d^2 \langle \beta W_{gen} \rangle}{d\theta^2}. \tag{5}$$

Our results identify critical regimes and show that during the phase transition, where the configuration entropy of the system decreases, the rates of change of the work and of the internal energy also decrease, while their curvatures diverge. The curvature of the internal energy, Eq. (4), can be interpreted both information-geometrically and computationally, as the difference between the curvature of the free entropy, captured by the Fisher information (the sensitivity of the system), and the curvature of the configuration entropy (the uncertainty of the system). This "computational balance" enhances the view of the "thermodynamic balance", shaped by the first law of thermodynamics in the context of quasi-static processes — the balance between the configuration entropy of the system, its internal energy and the work done on, or extracted from, the system.

The cross-disciplinary perspective adopted in this study allows us to consider a measure of the *thermodynamic efficiency of computation*, defined, for a given value of the control parameter θ, as the reduction in uncertainty (that is, the increase in the internal order) that resulted from an expenditure of work:

$$\eta \equiv \frac{-dS/d\theta}{d\langle \beta W_{gen} \rangle / d\theta} = \frac{-dS/d\theta}{\int_{\theta}^{\theta^*} F(\theta') d\theta'}, \tag{6}$$

where θ^* is the zero-response point for which small changes incur no work [5]. In light of Eq. (5), this ratio can be considered entirely in computational terms as the ratio of increasing order, obtained at θ, to the cumulative sensitivity incurred over a process from the current state θ to the state of perfect order, identified by the zero-response point θ^*.

Arguing that the system of self-propelled particles is a system performing collective computation, we focus on the balance between the *sensitivity* and the *uncertainty* of the computation. The collective motion has two disticnt phases (disordered motion or coherent motion), and we observe that the sensitivity and the uncertainty are balanced in each of these phases. However, at criticality, i.e., during a kinetic phase transition, this balance is broken. We find that, as the alignment strength between self-propelled particles, or the number of nearest neighbours affecting a particle's alignment, increases, the entropy decreases, while the work rate is positive. This means that the generation of order requires work to be spent. Specifically, we find that the ratio η of the generated order to the work rate, specified by Eq. (6), peaks precisely at the critical point. This indicates that the maximal thermodynamical efficiency of computation carried out by the system of self-propelled particles is highest during the phase transition.

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