# Exercise Session on Tensors

**Exercise 1.** (1) Let $e_1 = (1,0), e_2 = (0,1) \in \mathbb{R}^2$ and $d_1 = (1,0,0), d_2 = (0,1,0) \in \mathbb{R}^3$. Visualize the tensor $T = e_1 \otimes d_1 \otimes e_1 + e_2 \otimes d_2 \otimes e_2 \in \mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^2$ as a three-dimensional cube (or supermatrix). What entries does your supermatrix have?

(2) Define a group action on $HR^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^2$ that resembles the transposition of a matrix.

Now let $S^d(\mathbb{R}^n)$ denote the *d-th symmetric power* of $\mathbb{R}^n$; i.e., the space of real symmetric $n \times \cdots \times n$ ($d$-times) tensors.

(3) How does a symmetric rank-one tensor look like when written as a polynomial?

(4) Let $e_1, e_2, e_3$ be the three standard basis vectors in $\mathbb{R}^3$. Write the following tensors as polynomials.

$$a)\ T_1 = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} \in S^3(\mathbb{R}^3). \qquad b)\ T_2 = e_1^{\otimes 4} + e_2^{\otimes 4} + e_3^{\otimes 4} \in S^4(\mathbb{R}^3).$$

(5) Which element in $S^3(\mathbb{R}^2)$ could be described by the symbol $e_1 e_2 e_2$?

**Exercise 2.** Let $V$ be a vector space of some arbitrary field $\mathbb{K}$. We write $n := \dim_{\mathbb{K}} V$ and assume that $e_1, \ldots, e_n$ is a basis for $V$.

(1) Determine the ranks and border ranks of following tensors in $V \otimes V \otimes V$

$$a)\ e_1 \otimes e_2 \otimes e_1, \qquad b)\ e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_2 \otimes e_3.$$

(2) Define the $d$-th symmetric power $S^d(V)$ so that it matches the definition from Exercise 1 when $V = \mathbb{R}^n$. Can you give a coordinate-free definition?

(3) Determine the ranks and border ranks of following tensors in $S^3(V)$: *a)* $e_1e_1e_2$. *b)* $e_1e_2e_3$.

(4) Complete (and prove) sentences of the following type: 'In $V \otimes V \otimes V$ there exists a tensor of rank/border rank at least ...', 'There is no tensor with rank/border rank greater than ...'.

**Exercise 3.** Consider the following *bilinear map* $\psi : S^2(\mathbb{C}^2) \times S^2(\mathbb{C}^2) \to S^2(\mathbb{C}^2),\ (f, g) \mapsto f \cdot g$, where the product is defined as the usual product of two polynomials.

(1) Compute $\dim_{\mathbb{C}} S^2(\mathbb{C}^2)$.

(2) In which tensor space does $\psi$ live?

(3) Endow $S^2(\mathbb{C}^2)$ with a basis of your choice $\{\lambda_1, \lambda_2, \lambda_3\}$ and write the tensor $\psi$ in the rank-one basis $\{\lambda_i \otimes \lambda_j \otimes \lambda_k \mid 1 \le i, j, k \le 3\}$.

**Exercise 4.** In this exercise $\mathbb{P}^n$ denotes the $n$-dimensional complex projective space.

(1) Compute the dimension of $\sigma_2(\mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c)$ for a tuple $(a, b, c)$ of your choice. Can you generalize your result to other $(a, b, c)$?

(2) Can you give defining equations of $\sigma_2(\mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c)$? Generalize your result for other products of vector spaces.

(3) Can you find any equations of higher secant varieties?

**Exercise 5.** (1) What is the dimension of $S^e(S^d\mathbb{C}^n)$? How do you represent this space? In general, do you think it is a reducible or an irreducible representation of $\mathrm{GL}(\mathbb{C}^N)$?

(2) Do you know any (special) elements of $S^2(S^2(\mathbb{C}^2))$? Hint: Recall the formula to solve quadratic equations and think about how to express it in the present context.

(3) Can you decompose $S^2(S^2(\mathbb{C}^2))$?

**Exercise 6.** We write $V^{\otimes d} := V \otimes \cdots \otimes V$ ($d$ times).

(1) Show that $V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)$.

(2) Show that $V^{\otimes 3} \neq S^3(V) \oplus \Lambda^3(V)$. What is the missing piece? *Hint:* Consider the sequence $K \to S^2(V) \otimes V \xrightarrow{\iota} V^{\otimes 3}$, where $\iota$ is the natural embedding and $K = \ker \iota$.

## Exercise Session on Tropical semirings

The use of polymake for some of the exercises is strongly encouraged.

**Exercise 7.**

(1) Compute $A \odot B$, $A \oplus B$ and $A \oplus A^2 \oplus \cdots \oplus A^{100}$ for the following two matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 9 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 2 & 3 \\ 5 & 9 \end{pmatrix}.$$

(2) Let $D = (d_{ij}) \in \mathbb{R}_{\geq 0}^{n \times n}$ with $d_{ii} = 0$ for all $i$. Show that

$$d_{ik} \leq d_{ij} + d_{jk} \text{ for all } i, j, k \quad \Longleftrightarrow \quad D \oplus D = D.$$

Matrices with this property are referred to as representing *metric spaces.*

(3) Let $G$ be a positively weighted graph with adjacency matrix $D_G$ and $D_G^{n-1} = (d_{ij}^{(n-1)})_{i,j}$. Show

$$d_{ij}^{(n-1)} = \text{ shortest path from node } i \text{ to node } j.$$

(4) Let $G$ be a weighted graph with no negative cost circuit, i.e. negative weights and loops are allowed as long as the sum of all weights on the loops is non-negative. Let $D_G$ be its adjacency matrix and set $D_G^+ := D_G \oplus D_G^2 \oplus \cdots \oplus D_G^n = (d_{ij}^+)_{i,j}$. Show

$$d_{ij}^+ = \text{ shortest path from node } i \text{ to node } j.$$

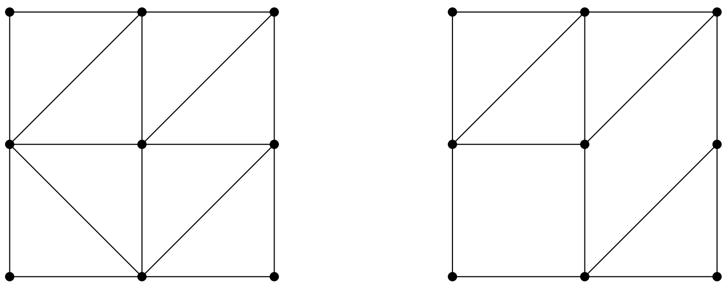
What goes wrong if $G$ has a negative cost circuit?

**Exercise 8.**

(1) Find all roots of the quintic $x^5 \oplus 1 \odot x^4 \oplus 3 \odot x^3 \oplus 6 \odot x^2 \oplus 10 \odot x \oplus 15$.
(2) Formulate and prove the Fundamental Theorem of Algebra for the tropical polynomial functions.

**Exercise 9.**

(1) Given five general points in $\mathbb{R}^2$, there exists a unique tropical quadric passing through these points. Compute and draw the quadratic curve through the points $(0, 5)$, $(1, 0)$, $(4, 2)$, $(7, 3)$ and $(9, 4)$.
(2) Find explicit tropical polynomials whose Newton subdivision looks as follows:



**Exercise 10.** The tropical $3 \times 3$-determinant is a piecewise-linear real-valued function on the nine-dimensional space of $3 \times 3$-matrices:

$$\text{tropdet}: \quad \mathbb{R}^{3\times 3} \to \mathbb{R}, \quad (a_{ij})_{i,j=1,2,3} \longmapsto \bigoplus_{\pi \in S_3} a_{1\pi(1)} \odot a_{2\pi(2)} \odot a_{3\pi(3)}.$$

Describe all regions of linearity of this function and their boundaries. What does it mean for a matrix to be tropically singular.

**Exercise 11.** Write a script that, using tropical geometry, implicitizes the image of a tuple of rational functions

$$\Phi: \quad \mathbb{C} \longrightarrow \mathbb{C}^2, \quad t \longmapsto (\Phi_1(t), \Phi_2(t)).$$