Preparation Exercises for the Chow Lectures

Franco Giovenzana, Kathlén Kohn

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1 Hermitian and Kähler forms

Exercise/Definition 1. Let V be a complex vector space of finite dimension. Show that the following three sets are canonically isomorphic:

(i) the set of *Hermitian forms on* V

$$\left\{ h : V \times V \to \mathbb{C} \;\middle|\; \begin{array}{l} \forall \alpha, \beta \in \mathbb{C}\, \forall u_1, u_2, v \in V : \\ h(\alpha u_1 + \beta u_2, v) = \alpha h(u_1, v) + \beta h(u_2, v), \\ h(v, u_1) = \overline{h(u_1, v)} \end{array} \right\},$$

(ii) the set of symmetric bilinear forms on V that are invariant under multiplication by i

$$\left\{ q : V \times V \to \mathbb{R} \;\middle|\; \begin{array}{l} \forall \alpha, \beta \in \mathbb{R}\, \forall u_1, u_2, v \in V : \\ q(\alpha u_1 + \beta u_2, v) = \alpha q(u_1, v) + \beta q(u_2, v), \\ q(v, u_1) = q(u_1, v),\ q(iu_1, iv) = q(u_1, v) \end{array} \right\},$$

(iii) the set of *exterior 2-forms on* V that are invariant under multiplication by i

$$\left\{ \omega : V \times V \to \mathbb{R} \;\middle|\; \begin{array}{l} \forall \alpha, \beta \in \mathbb{R}\, \forall u_1, u_2, v \in V : \\ \omega(\alpha u_1 + \beta u_2, v) = \alpha \omega(u_1, v) + \beta \omega(u_2, v), \\ \omega(v, u_1) = -\omega(u_1, v),\ \omega(iu_1, iv) = \omega(u_1, v) \end{array} \right\}.$$

The isomorphism is set up by:

$$\begin{array}{ll} q = \Re(h), & \omega = -\Im(h) \text{ is called } \textit{Kähler form of } h, \\ h(u, v) = q(u, v) - iq(iu, v), & \omega(u, v) = q(iu, v) \\ h(u, v) = -\omega(iu, v) - i\omega(u, v), & q(u, v) = -\omega(iu, v). \end{array}$$

Moreover, show that h is positive definite if and only if q is positive definite. In this case, h is called *Hermitian metric*.

2 Almost Complex Structures

Definition. An *almost complex structure* on a finite-dimensional real vector space V is an endomorphism

$$I : V \longrightarrow V \qquad \text{such that} \qquad I^2 = -\mathrm{id}_V.$$

Exercise 2. A real vector space V with an almost complex structure I is a complex vector space via the complex scalar multiplication

$$(a+ib)v := av + bI(v) \qquad \text{for } a, b \in \mathbb{R}, v \in V.$$

Remark. This shows that real vector spaces with almost complex structures and complex vector spaces are equivalent notions. In particular, an almost complex structure can only exist on an even-dimensional real vector space.

Exercise 3. Let V be a finite-dimensional real vector space with an almost complex structure $I : V \to V$. We extend I to a $\mathbb{C}$-linear map on the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ via

$$\begin{aligned} I : V \otimes_{\mathbb{R}} \mathbb{C} &\longrightarrow V \otimes_{\mathbb{R}} \mathbb{C}, \\ z \cdot v &\longmapsto z \cdot I(v) \quad \text{ for } z \in \mathbb{C}, v \in V. \end{aligned}$$

a. Show that the eigenvalues of I are $\pm i$, giving the decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ denotes the eigenspace of i and $V^{0,1}$ the eigenspace of $-i$. Moreover, complex conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ induces an $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.

b. Show that

$$\begin{aligned} \Psi : (V \otimes_{\mathbb{R}} \mathbb{C})^* = \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^* \otimes_{\mathbb{R}} \mathbb{C}, \\ \varphi &\longmapsto \Psi(\varphi) : \begin{cases} V & \to \mathbb{C}, \\ v & \mapsto \varphi(v \otimes 1) \end{cases} \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism.

c. Show that the decomposition $(V \otimes_{\mathbb{R}} \mathbb{C})^* = (V^{1,0})^* \oplus (V^{0,1})^*$ induces a decomposition

$$V^* \otimes_{\mathbb{R}} \mathbb{C} = \Psi\left((V^{1,0})^*\right) \oplus \Psi\left((V^{0,1})^*\right)$$

into $\mathbb{C}$-*linear* and $\mathbb{C}$-*antilinear* maps from V to $\mathbb{C}$, i.e.,

$$\begin{aligned} \Psi\left((V^{1,0})^*\right) &= \{\psi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \psi(I(v)) = i\psi(v) \text{ for all } v \in V\}, \\ \Psi\left((V^{0,1})^*\right) &= \{\psi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \psi(I(v)) = -i\psi(v) \text{ for all } v \in V\}. \end{aligned}$$

Remark. Denoting $W^{j,l} := \Psi\left((V^{j,l})^*\right)$, we get that

$$\bigwedge^k \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \bigoplus_{p+q=k} \bigwedge^p (W^{1,0}) \otimes \bigwedge^q (W^{0,1}).$$

3 Singular (Co)Homology

Definition. Let X be a topological space.

(i) For $k \in \mathbb{Z}_{\geq 0}$, let $C_k(X)$ be the free abelian group generated by the continuous maps from the simplex Δ^k of dimension k to X. For $k \in \mathbb{Z}_{>0}$, let

$$\partial_k : C_k(X) \longrightarrow C_{k-1}(X),$$
$$\sigma \longmapsto \sum_{i=0}^{k} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_k]}$$

where $[v_0, \dots, \widehat{v_i}, \dots, v_k]$ denotes the face of Δ^k (which has vertices $v_0, \dots, v_k$) obtained by removing the i-th vertex. Moreover, let $\partial_0 : C_0(X) \to \{0\}, \sigma \mapsto 0$. We have that $\partial_k \circ \partial_{k+1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

$$\dots \xrightarrow{\partial_4} C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \{0\}$$

is called the *singular chain complex*. The *k-th homology group of X* is $H_k(X) := \ker(\partial_k)/\mathrm{im}\,(\partial_{k+1})$.

(ii) Fix an abelian group G. For $k \in \mathbb{Z}_{\geq 0}$, define

$$\delta_k : \mathrm{Hom}(C_k(X), G) \longrightarrow \mathrm{Hom}(C_{k+1}(X), G),$$
$$\varphi \longmapsto \varphi \circ \partial_{k+1}.$$

Moreover, let $\delta_{-1} : \{0\} \to \mathrm{Hom}(C_0(X), G), 0 \mapsto 0$. Since $\delta_k \circ \delta_{k-1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$, we get the *singular cochain complex*

$$\{0\} \xrightarrow{\delta_{-1}} \mathrm{Hom}(C_0(X), G) \xrightarrow{\delta_0} \mathrm{Hom}(C_1(X), G) \xrightarrow{\delta_1} \mathrm{Hom}(C_2(X), G) \xrightarrow{\delta_2} \dots.$$

The *k-th cohomology group of X with coefficients in G* is $H^k(X, G) := \ker(\delta_k)/\mathrm{im}\,(\delta_{k-1})$.

Exercise 4. Let $X = \{x\}$ be a singleton. Show the following:

a. $C_k(X) \simeq \mathbb{Z}$ for all $k \in \mathbb{Z}_{\geq 0}$,

b. $\partial_k = 0$ for $k \in \mathbb{Z}_{>0}$ odd, and ∂_k is an isomorphism for $k \in \mathbb{Z}_{>0}$ even,

c. $H_0(X) \simeq \mathbb{Z}$, and $H_k(X) = \{0\}$ for $k \in \mathbb{Z}_{>0}$,

d. $H^0(X, \mathbb{Z}) \simeq \mathbb{Z}$, and $H^k(X, \mathbb{Z}) = \{0\}$ for $k \in \mathbb{Z}_{>0}$.

Exercise 5. Show that $H_0(\mathbb{R}^1 \setminus \{0\}) \simeq \mathbb{Z} \times \mathbb{Z} \simeq H^0(\mathbb{R}^1 \setminus \{0\}, \mathbb{Z})$ and that $H_0(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{Z} \simeq H^0(\mathbb{R}^2 \setminus \{0\}, \mathbb{Z})$.

Remark. The advantage of using cohomology instead of homology is the existence of the *cup product* which makes $H^*(X, R) := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^k(X, R)$ into a graded ring for a commutative ring R. The cup product is a bilinear operation $H^i(X, R) \times H^j(X, R) \to H^{i+j}(X, R)$ induced by

$$\mathrm{Hom}(C_i(X), R) \times \mathrm{Hom}(C_j(X), R) \longrightarrow \mathrm{Hom}(C_{i+j}(X), R),$$
$$(\varphi, \psi) \longmapsto \left(\sigma \mapsto \varphi(\sigma|_{[v_0,\dots,v_i]}) \cdot \psi(\sigma|_{[v_i,\dots,v_{i+j}]})\right).$$