Convexity Day

MPI Leipzig, September 9, 2019

Some hands-on activities suggested by **Bernd Sturmfels**

(1) Consider the set $C = \{x \in \mathbb{R}^n : f(x) \leq 1\}$ where $f(x)$ is a polynomial in n real variables. How can you test whether C is compact? How to test whether C is convex?

(2) Is there an algorithm for testing whether a convex set C given as in (1) is a *zonoid.*

(3) Give an example of a polynomial f in $n = 3$ variables of degree four such that the set C in (1) is a zonoid. Do these f form a semialgebraic subset in $\mathbb{R}[x, y, z]_{\leq 4} \simeq \mathbb{R}^{35}$?

(4) Let T be the triangle with vertices in $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$ in $\mathbb{R}^3$ and consider the unit ball $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. Is the Minkowski sum $T + B$ a *basic semialgebraic* set? Give a nice description of $T + B$ by polynomial inequalities.

(5) Redo Problem (4) with T replaced by a random ellipse in the plane $\{x + y + z = 1\}$.

(6) Redo problems (4) and (5) with B replaced by a random ellipsoid in $\mathbb{R}^3$.

(7) In Problems (4), (5) and (6), determine the coefficients of the polynomial function $x \mapsto \mathrm{vol}(T + xB)$. Can you use exact arithmetic in writing down these coefficients?

(8) Are the convex sets in (4), (5) and (6) *spectrahedra*? How about their convex duals?

(9) Pick three random ellipsoids in $\mathbb{R}^3$. Determine their convex hull and its dual.

(10) The logo of the MPI Nonlinear Algebra Group is the spectrahedron

$$\mathcal{S} \;=\; \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}.$$

Compute the *projection body* of $\mathcal{S}$. Is that projection body a (basic) semialgebraic set?

(11) Compute the *intersection body* of the octahedron $P = \mathrm{conv}(\pm e_1, \pm e_2, \pm e_3\}$ in $\mathbb{R}^3$.

(12) Study the convex hull of the trigonometric space curve $(\cos(\theta), \cos(2\theta), \sin(3\theta))$.

(13) Describe the *normal cycles* of all convex bodies seen on this page. Yes, use polynomials!

(14) True or false: Given a (centrally symmetric) convex body that is semialgebraic then both its projection body and its intersection body are also semialgebraic.

(15) Generalize the spectrahedron in (10) by introducing positive real parameters a, b, c:

$$\mathcal{S}_{abc} \quad = \quad \left\{ (x, y, z) \in \mathbb{R}^3 \,:\, \begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix} \succeq 0 \right\}.$$

Prove that the volume of $\mathcal{S}_{abc}$ is equal to $(1/2)abc \cdot \pi^2$. How about the surface area?

(16) A function $p : \mathbb{R}^2 \to \mathbb{R}$ is *convex* if its Hessian determinant is nonnegative at every point. Study this property for functions defined by binary sextics

$$p(x, y) \quad = \quad ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6.$$

Show that the following set is a semialgebraic convex cone

$$\mathcal{C} \quad = \quad \big\{ (a, b, c, d, e, f, g) \in \mathbb{R}^7 : \text{the function } p(x, y) \text{ is convex} \big\}.$$

What are the faces of $\mathcal{C}$? What is the degree (and defining polynomial) of its boundary?

(17) Why are *Lorentzian polynomials* relevant for convex geometry?

(18) Why does Antonio Lerario like the *kinematic formulas*?

(19) What are all the *tensor valuations* of the spectrahedron $\mathcal{S}$ in (10)?

(20) What can you say about the convex hull of the rotation group SO(3) in the space of 3×3-matrices? What are the faces of this convex body? How to describe the dual convex body? How would you compute features like volume, algebraic boundary, singularities, normal cycle for these 9-dimensional convex bodies? Why does it matter?