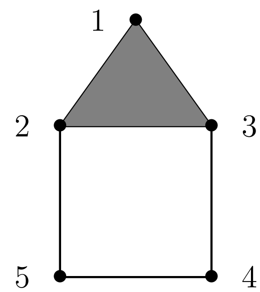
EXERCISES FOR "RANDOMNESS AND SYZYGIES", MPI LEIPZIG SUMMER SCHOOL

DANIEL ERMAN

Theorem 0.1 (Hochster's Formula)**.** *Let Δ be a simplicial compelx with n vertices. Let K be any field and let $S = K[x_1, \dots, x_n]$. Then:*

$$\beta_{i,j}(S/I_\Delta) = \sum_{\alpha \in \binom{[n]}{j}} \dim \widetilde{H}^{j-i-1}(\Delta|_\alpha; K).$$

(1) **Stanley-Reisner Theory:** Let Δ be the simplicial complex on 5 vertices consisting of $\{1,2,3\}, \{2,5\}, \{4,5\}, \{3,4\}$ and all subfaces.



   (a) Write down I_Δ. (Remember that I_Δ is generated by the minimal non-faces of Δ.
   (b) Use Hochster's formula to compute $\beta_{i,j}(S/I_\Delta)$ for all i and j. (Hint: you will only get a nonzero answer for $\beta_{0,0}, \beta_{1,2}, \beta_{2,3}, \beta_{2,4}$ and $\beta_{3,5}$.)
   (c) Bonus: Compute a primary decomposition of I_Δ. Using this, explain how the above graph is really a picture of the vanishing locus $V(I_\Delta)$.

(2) **Expected Betti Numbers of a Random Ideal:** Let $\Delta \in \Delta(n,p)$ be a random flag complex where $p \gg \frac{1}{n}$. Let's show that the expected value of $\beta_{i,i+2}(I_\Delta)$ goes to infinity.
   (a) Use Hochster's formula to show that $\beta_{i,i+2}(\Delta)$ is at least the number of subgraphs on $i+2$ vertices which consist of a single cycle of size $i+2$.
   (b) Show that, for a given $\alpha \in \binom{[n]}{i+2}$ the probability that $\Delta|_\alpha$ is a single cycle of size $i+2$ is at least p^{i+2}.
   (c) Combine parts (a) and (b) to show that, for any fixed i, the expected value of $\beta_{i,i+2}$ goes to ∞ as $n \to \infty$.

(3) **Philosophy:** What areas in your own field might be amenable to seeking out a heuristics to make predictions where computation is infeasible? Discuss with others!

(4) Go look at syzygydata.com. Check out the cool Betti tables that Juliette Bruce and Jay Yang and others computed and put up there. Look for patterns and make conjectures and prove theorems. (This one might be difficult.)