TROPICAL GEOMETRY, P-ADICS, PROBABILITY AND APPLICATIONS

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Abstract. Some exercises leading to open questions on this theme, intended for early-years PhD students. The section on Lattices, probability measures and Fourier was compiled by Yassine El Maazouz.

1. Lattices, probability measures and Fourier theory

Definition 1.1. Let $d \geq 1$. We call lattice in $\mathbb{Q}_p^d$ a rank d $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^d$.

Exercise 1.2. Show that $\mathrm{GL}_d(\mathbb{Q}_p)$ acts transitively on the set of lattices. What is the stabilizer of the lattice $\mathbb{Z}_p^d$? Deduce that stabilizer of a general lattice L.

Exercise 1.3. Show that for any matrix $A \in \mathbb{Q}_p^{d\times d}$, there exists $U, V \in \mathrm{GL}_d(\mathbb{Z}_p)$ such that UAV is diagonal. Can you give an algorithm to compute U and V?

Exercise 1.4. Show that the stabalizer of a lattice L in $\mathrm{GL}_d(\mathbb{Q}_p)$ is a compact subgroup of $\mathrm{GL}_d(\mathbb{Q}_p)$. Describe the maximal compact subgroups of $\mathrm{GL}_d(\mathbb{Q}_p)$.

Exercise 1.5. Let $A \in \mathrm{GL}_d(\mathbb{Q}_p)$. Show that there exists $U \in \mathrm{GL}_d(\mathbb{Z}_p)$ such that AU is lower triangular. Give an algorithm to compute U.

Exercise 1.6. Show that for each lattice L there exists a unique maximal (in inclusion) lattice $L' \subset L$ such that $L' = D\,\mathbb{Z}_p^d$ where D is a diagonal matrix. Show that there is a unique minimal lattice L'' containing L such that L'' is diagonal. Give algorithms to compute these lattices.

Exercise 1.7. Let L be the lattice represented by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & p & 0 \\ 1 & p^{-1} & p^{-1} & p^2. \end{bmatrix}$$

Find U, V such that UAV is diagonal. Compute L' and L'' defined in the previous exercise.

Exercise 1.8. Give an example of a non-trivial unitary continuous character $\chi : \mathbb{Q}_p \to \mathbb{C}^\times$.

Exercise 1.9. Let G be a compact topological group. Show that any continuous character of G is necessarily unitary. Deduce that all continuous characters of $\mathbb{Q}_p$ are unitary.

Exercise 1.10. Let $\chi : \mathbb{Q}_p \to \mathbb{C}^\times$ be a non-trivial unitary continuous character. Describe the group $G = \chi^{-1}(\{1\})$.

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Exercise 1.11. Let μ be the unique Haar measure on $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$ and $\chi : (\mathbb{Q}_p, +) \to \mathbb{C}^\times$ a non-trivial character. Compute the following integral

$$\phi_\mu(u) = \int_{\mathbb{Z}_p} \chi(ux) d\mu(x), \quad u \in \mathbb{Z}_p \,.$$

The function ϕ_μ is the characteristic function of the measure μ. What can you say for the multivariate case?

2. Tropical basis

Recall the following (quote from [EKL06])

> **Remark 2.2.8.** Let I be the ideal in $\mathbb{k}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ defining X. Then trivially $\mathcal{T}(X) \subset \mathcal{T}(f)$ for every $f \in I$. Speyer and Sturmfels [23, Thm. 2.1] have shown that
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> $$\mathcal{T}(X) = \bigcap_{f \in I} \mathcal{T}(f).$$
>
> Furthermore, they describe in [23, Cor. 2.3] that the intersection can be taken over just those f in a (finite) universal Gröbner basis for I. Hence a tropical variety is always the intersection of a finite number of tropical hypersurfaces, each of which has an explicit description as a Γ-rational polyhedral set from Theorem 2.1.1. Their approach can be developed into an alternative proof of Theorem 2.2.5.

A tropical basis of I is a finite generating set $F \subset I$ such that $\bigcap_{f \in F} T(f) = T(X)$. For some ideals, the obvious generators form a tropical basis.

Exercise 2.1. Show that if $f \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $I = \langle f \rangle$, then $F = \{f\}$ is a tropical basis for I.

The following is a classic example, with interesting connections with phylogenetic trees.

Exercise 2.2. Show that the quadratic Plücker relations form a tropical basis for the Grassmanian$(n, 2)$.

For most ideals, finding the tropical basis and understanding it can be difficult. For an early tutorial by Bernd Sturmfels with open questions for grad students, see `https://math.berkeley.edu/~bernd/tropical/sec3.pdf`. General algorithms have been proposed [HT09], most recently [JS18, MR20].

Here are some ideals of interest for which the tropical basis is not yet known. A good start would be to revisit these problems while testing out the latest understandings and algorithms for tropical bases.

Open Problem 2.3. Understand the tropical basis for the Plücker ideal $I_{d,n}$. This ideal is generated by all $d \times d$ minors of a generic $d \times n$ matrix with coefficients in the field k. See [JS18, Example 9] for upperbounds. If $k = \mathbb{Q}_p$, does the result simplify in some way?

Open Problem 2.4. Find a tropical basis for the variety of commuting tropical matrices. This is the variety generated by n^2 polynomials obtained from $AB - BA = 0$

where A, B are generic $n \times n$ matrices over a field k. If $k = \mathbb{Q}_p$, does the result simplify in some way? See Chapter 5 of `http://sites.williams.edu/10rem/files/2016/07/Ralph-Morrison-Dissertation.pdf` for more details and initial computations.

3. Random tropicalized polynomials

Recall the following. Let $k = \mathbb{Q}_p$. Take a random polynomial in $f \in k[x, y]$:

$$f(x, y) = \sum_{(i,j)\in P} G_{ij} x^i y^j \tag{1}$$

where P is a lattice polytope in $\mathbb{N}^2$, and G_{ij} are p-adic Gaussians. Now, tropicalize f, we get a random tropical polynomial

$$f^{trop}(x, y) = \bigoplus_{(i,j)\in P} C_{ij} \odot x^{\odot i} y^{\odot j}. \tag{2}$$

with coefficients $C_{ij} = \mathtt{val}(G_{ij})$.

The general idea is to use properties of p-adic Gaussians to say something about f^{trop}, and then use tropical algebraic geometry to deduce something about f.

3.1. Systems of random p-adic polynomials.

3.1. Systems of random p-adic polynomials. Here is the abstract of the paper [Eva06], titled *The expected number of zeros of a random system of p-adic polynomials.*

Abstract

We study the simultaneous zeros of a random family of d polynomials in d variables over the p-adic numbers. For a family of natural models, we obtain an explicit constant for the expected number of zeros that lie in the d-fold Cartesian product of the p-adic integers. Considering models in which the maximum degree that each variable appears is N, this expected value is

$$p^{d\lfloor \log_p N \rfloor}(1 + p^{-1} + p^{-2} + \cdots + p^{-d})^{-1}$$

for the simplest such model.

Open Problem 3.1. Find a tropical proof of the main theorem of [Eva06].

3.2. Random tropical plane curves. This paper enumerates tropical plane curves of degree d by their genuses [BJMS15]. Each plane curve comes from a tropical polynomial in two variables of degree d. Here is the abstract.

> **Abstract**
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> We study the moduli space of metric graphs that arise from tropical plane curves. There are far fewer such graphs than tropicalizations of classical plane curves. For fixed genus g, our moduli space is a stacky fan whose cones are indexed by regular unimodular triangulations of Newton polygons with g interior lattice points. It has dimension $2g+1$ unless $g \leq 3$ or $g=7$. We compute these spaces explicitly for $g \leq 5$.

Open Problem 3.2. What happens if we tropicalize a $\mathbb{Q}_p$-adic polynomial with i.i.d coefficients? For example, find the distribution of its genus.

The main part of the computational difficulty of [BJMS15] is that they are looking at *unimodular* regular subdivisions. Here is a brief background (for more, see [Zie12]).

Let P be a lattice polytope. For a canonical example, take the dilated triangle $d \cdot \Delta_2 \subset \mathbb{R}^2$. The vertices of this triangle are $(0,0), (0,d)$ and $(d,0)$. A regular subdivision of P is obtained by lifting each lattice point $(i,j) \in P$ to some height c_{ij}, take the lower convex hull of the lifted points $\{(i,j,c_{ij}) \in \mathbb{R}^3\}$, and then project it backdown to $\mathbb{R}^2$. Sometimems people take the upper convex hull, this is just a matter of convention, like the max/min convetion in tropical geometry. See here for some pictures `http://www.rambau.wm.uni-bayreuth.de/Diss/diss_MASTER/node9.html`. This regular subdivision is dual to the tropical hypersurface of the polynomial

$$f^{trop}(x,y) = \bigoplus_{(i,j)\in P} c_{ij} \odot x^{\odot i} y^{\odot j}.$$

Thus, random tropical polynomials give rise to random subdivisions.

Exercise 3.3. Give a simple recipe to generate a random regular subdivision of any lattice polytope P.

A regular subdivision is a triangulation if each maximal cell is a triangle. It is unimodular if there is no cell with interior lattice points. That is, each cell in the regular subdivision is a triangle, whose only lattice points are its three vertices. See [HPPS21] for nice pictures, precise definitions and a list of what's known, what's not. Unimodular subdivisions correspond to smooth tropical curves.

Open Problem 3.4. How to easily generate a *random* unimodular triangulation of $d \cdot \Delta_2$? Of general dilated simplces $d \cdot \Delta_n$? Of the dilated cube $d \cdot [0,1]^n$?

Exercise 3.5. Suppose f^{trop} is obtained by tropicalizing (1) with G_{ij} i.i.d p-adic Gaussians for $P = d \cdot \Delta_2$. What is the expected number of cells of the corresponding regular subdivision? What happens to this number when $d \to \infty$? What does this say about the tropical curve? What does this say about the original p-adic polynomial f?

Things are more interesting when the G_{ij}'s are not i.i.d.

Open Problem 3.6. Let $G \in \mathbb{Q}_p^{d\times d}$ a random $d \times d$ matrix whose column vectors are i.i.d vectors drawn from a p-adic Gaussian distribution with lattice L. Define the random quadratic polynomial

$$f(x,y) = \sum_{i,j=1}^{d} G_{ij} x^i y^j$$

and let f^{trop} be its tropicalization, Δ_f be the corresponding regular subdivision. What can we say about Δ_f? For example, answer the same questions as those in the above exercise. Note that the previous exercise corresponds to the special case where the lattice L is the standard lattice.

References

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