Information, Interest Rates and Geometry

Dorje C. Brody

Department of Mathematics,
Imperial College London, London SW7 2AZ
www.imperial.ac.uk/people/d.brody

(Based on work in collaboration with Lane Hughston & Robyn Friedman)

Brief overview of interest rate theory

Dynamical models for interest rates suffer from the fact that it is difficult to isolate the independent degrees of freedom.

The question is: which ingredients in the determination of an interest rate model can and should be specified independently and exogenously?

A related issue, important for applications, is the determination of an appropriate data set for the specification of initial conditions.

This is the so-called 'calibration problem'.

Traditionally interest rate models have tended to focus either on discount bonds or on rates.

Depending on which choice is made, the resulting models take different forms, and hence have a different feel to them.

Fundamentally, however, it should not make any difference whether a model is based on bonds or rates.

To develop this point further let us introduce some notation.

Let time 0 denote the present.

We write P_{tT} for the value at time t of a discount bond that matures to deliver one unit of currency at time T.

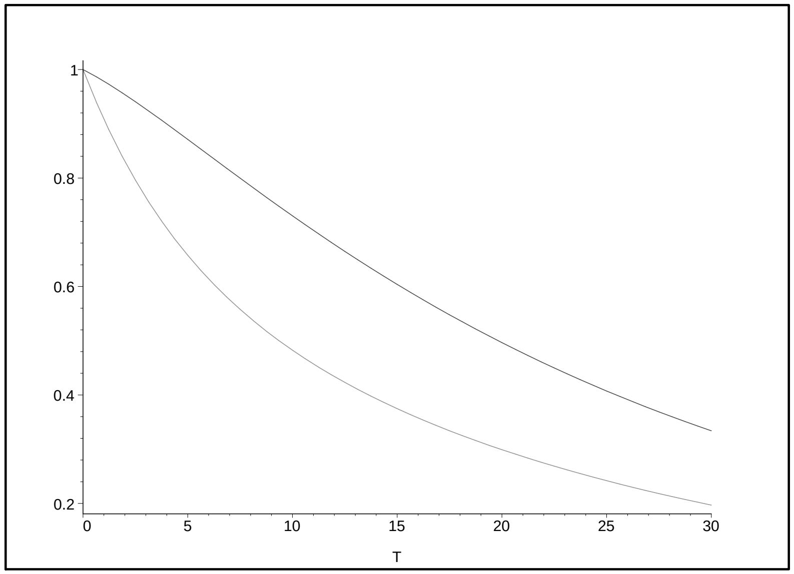

Figure 1: Typical discount bond prices P_{0T}.

The associated continuously compounded rate R_{tT} is defined by

$$P_{tT} = \exp\left(-(T-t)R_{tT}\right). \tag{1}$$

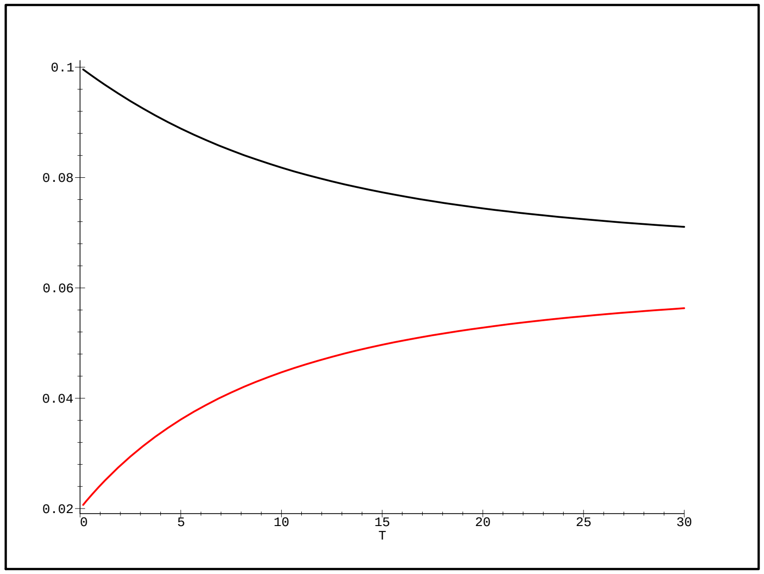

Figure 2: Typical yield curves R_{0T}.

The dynamics of R_{tT} and P_{tT} look different from one another, even if the underlying model is the same.

Once we have the discount bond system, the associated rates can be directly constructed.

For instance, the short (overnight) rate is defined by

$$r_t = -\frac{\partial}{\partial T} \ln P_{tT} \bigg|_{T=t} . \tag{2}$$

Dynamic models for the short rate

The model is defined with respect to a given probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}$ and a standard multidimensional Brownian motion $\{W_t^\alpha\}$ $(\alpha = 1, 2, \cdots, n)$, where n is possibly infinite.

Here $\mathbb{Q}$ denotes the "risk-neutral" measure.

The independent degree of freedom is given by

- the specification of the short rate $\{r_t\}$ as an essentially arbitrary Ito process on $(\Omega, \mathcal{F}, \mathbb{Q})$.

The model for the discount bonds is

$$P_{tT} = \mathbb{E}\left[\exp\left(-\int_t^T r_s \mathrm{d}s\right) \bigg| \mathcal{F}_t\right] . \tag{3}$$

An advantage of this approach is that the process $\{r_t\}$ can be specified independently and exogenously.

There are three disadvantages to this approach:

- the model is specified implicitly: the conditional expectation is generally difficult to calculate.
- the initial term structure P_{0T} is not fed in directly.
- it is difficult to generate a positive model for $\{r_t\}$ without making the model somewhat artificial.

Term structure density approach

This approach has the virtue of eliminating undesirable features of rate-based modelling while retaining desirable features.

Consider first the initial discount function P_{0T}.

Positivity of nominal rates implies that the discount function P_{0T} is decreasing in the maturity variable T.

A common sense argument shows that a bond with infinite maturity has no value.

Thus P_{0T} can be thought of as defining a right-side cumulative distribution function on the positive real line $\mathbb{R}_+$.

In particular, $\rho_0(T) = -\partial_T P_{0T}$ defines a density function over $\mathbb{R}_+$.

Hence the positive interest term structure implies the existence of a random variable X on $(\Omega, \mathcal{F}, \mathbb{Q})$ such that we have

$$P_{0T} = \mathbb{Q}(X \geq T). \tag{4}$$

By consideration of the square-root map

$$\rho_0(T) \to \xi(T) = \sqrt{\rho_0(T)} \tag{5}$$

we see that the system of admissible term structure is isomorphic to the convex space $D(\mathbb{R}_+)$ of smooth density functions on the positive real line.

The "distance" between a pair of term structures can thus be measured by

$$\phi(\rho_1, \rho_2) = \cos^{-1} \int_0^\infty \xi_1(T)\xi_2(T)\mathrm{d}T. \tag{6}$$

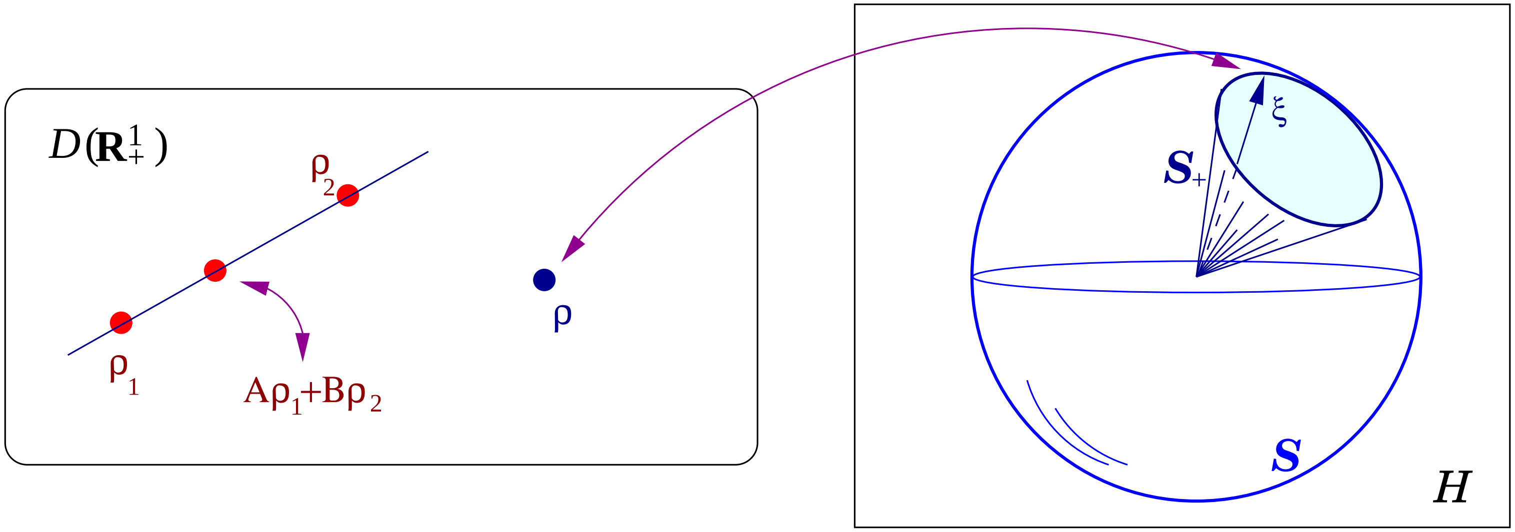

Figure 3: The system of admissible term structures.

Dynamics of the term-structure density

For the general discount bond dynamics, let us write

$$\mathrm{d}P_{tT} = r_t P_{tT}\mathrm{d}t + \Sigma_{tT}\mathrm{d}W_t, \tag{7}$$

where Σ_{tT} is the *absolute volatility processes* for a bond with maturity T.

In order to extend the analysis it is convenient to write

$$B_{tx} = P_{t,t+x} \tag{8}$$

where x is the time left until maturity.

Thus B_{tx} is the value at time t of a discount bond that has x years left to mature.

It follows that

$$\rho_t(x) = -\frac{\partial}{\partial x} B_{tx} \tag{9}$$

is a measure-valued process in the sense that for each value of t the random function $\rho_t(x)$ satisfies $\rho_t(x) > 0$ and the normalisation condition

$$\int_0^\infty \rho_t(x)\mathrm{d}x = 1. \tag{10}$$

The dynamics of B_{tx} is determined by

$$\mathrm{d}B_{tx} = (\mathrm{d}P_{tT})|_{T=t+x} + \frac{\partial}{\partial x} B_{tx}\mathrm{d}t, \tag{11}$$

and hence

$$\mathrm{d}B_{tx} = (r_t B_{tx} + \partial_x B_{tx})\,\mathrm{d}t + \Sigma_{t,t+x}\mathrm{d}W_t, \tag{12}$$

where $\partial_x = \partial/\partial x$.

Differentiating this expression with respect to x, we obtain

$$\mathrm{d}\rho_t(x) = (r_t\rho_t(x) + \partial_x\rho_t(x))\,\mathrm{d}t + \omega_{tx}\mathrm{d}W_t, \tag{13}$$

where $\omega_{tx} = -\partial_x \Sigma_{t,t+x}$.

Before proceeding further, we remark that the short rate satisfies the relation:

$$r_t = -\int_0^\infty \rho_t(x)\partial_x \ln \rho_t(x)\mathrm{d}x. \quad (14)$$

In other words, r_t is minus the expectation of the gradient of the log-likelihood function.

Let us now examine more closely the volatility term ω_{tx}.

Because $\rho_t(x)$ must remain positive for all values of x, the coefficient of $\mathrm{d}W_t$ must be of the form

$$\omega_{tx} = \rho_t(x)\sigma_{tx} \quad (15)$$

such that

$$\sigma_{tx} = \nu_{tx} - \mathbb{E}_\rho[\nu_{tx}], \quad (16)$$

where ν_{tx} is an exogenously specifiable unconstrained process.

As a consequence, we deduce

$$\frac{\mathrm{d}\rho_t(x)}{\rho_t(x)} = (r_t + \partial_x \ln \rho_t(x))\mathrm{d}t + \sigma_{tx}\mathrm{d}W_t. \tag{17}$$

This is the general dynamical equation satisfied by the term structure density process.

Hilbert space dynamics for term structures

Let us consider how we transform to the Hilbert space representation for density functions.

Denote by ξ_{tx} the process for the square-root likelihood function, defined by

$$\rho_t(x) = \xi_{tx}^2. \tag{18}$$

It follows by Ito's lemma that

$$\mathrm{d}\rho_t(x) = 2\xi_{tx}\mathrm{d}\xi_{tx} + (\mathrm{d}\xi_{tx})^2, \tag{19}$$

and hence

$$\mathrm{d}\xi_{tx} = \left(\partial_x\xi_{tx} + \tfrac{1}{2}r_t\xi_{tx} - \tfrac{1}{8}\xi_{tx}\sigma_{tx}^2\right)\mathrm{d}t + \tfrac{1}{2}\xi_{tx}\sigma_{tx}\mathrm{d}W_t. \tag{20}$$

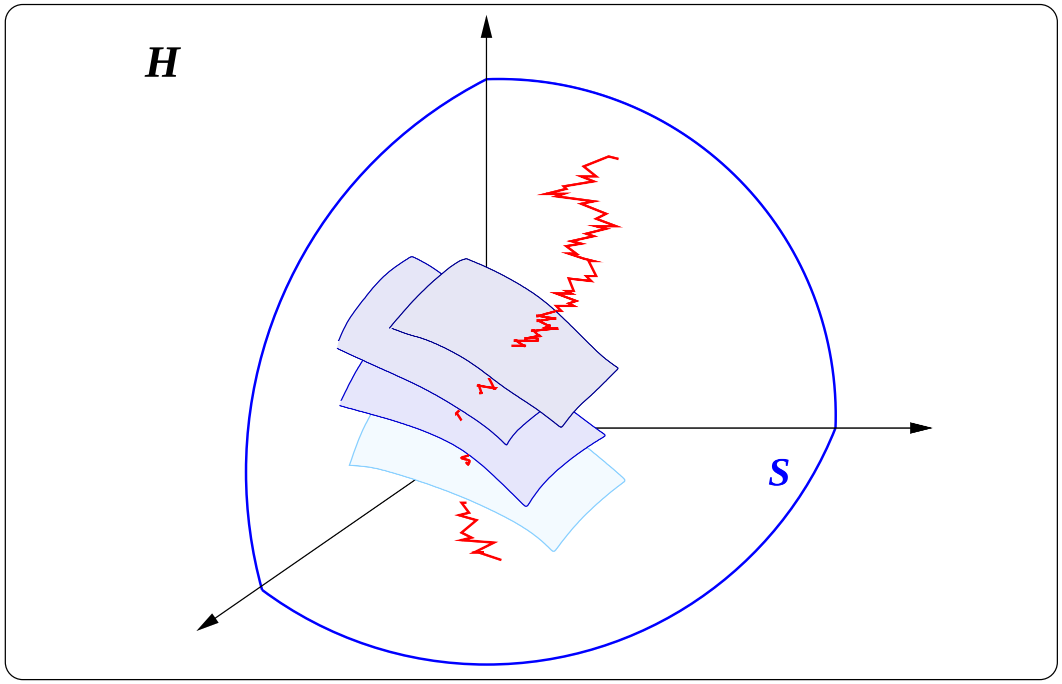

Figure 4: Interest rate dynamics on Hilbert space.

We would like to interpret the Hilbert space dynamics in a geometrical fashion.

For this purpose we find it expedient to introduce the Dirac notation to signify Hilbert space operations

Thus if the function $\psi(x)$ is an element of $\mathcal{H} = L^2(\mathbb{R}^1_+)$, we denote it by $|\psi\rangle$, and if $\varphi(x)$ belongs to the dual Hilbert space $\mathcal{H}^*$ we denote this by $\langle\varphi|$.

Their inner product is thus written

$$\langle\psi|\varphi\rangle = \int_0^\infty \psi(x)\varphi(x)\mathrm{d}x. \tag{21}$$

Now suppose $\xi(x)$ is a positive function.

In that case, the derivative ∂_x can be thought of as a linear operator $\hat{D}$ on $\mathcal{H}$, and we have an endomorphism given by

$$|\xi\rangle \to \hat{D}|\xi\rangle \tag{22}$$

providing that $|\xi\rangle$ lies in the domain of $\hat{D}$.

We can now tentatively interpret, in the language of Hilbert space geometry, the terms appearing in the drift in the dynamical equation.

Let us begin by noting first that (14) can be rewritten in the form

$$\int_0^\infty \xi_{tx}\partial_x\xi_{tx}\mathrm{d}x = -\tfrac{1}{2}\,r_t. \tag{23}$$

This allows us to interpret the short term interest rate process r_t in terms of the mean of the symmetric part of the operator $\hat{D}$ in the state $|\xi_t\rangle$.

In particular, we have

$$\frac{\langle \xi_t | \hat{D} | \xi_t \rangle}{\langle \xi_t | \xi_t \rangle} = -\tfrac{1}{2}\, r_t. \tag{24}$$

Therefore, if we let $\hat{D}_{\rm s}$ denote the symmetric part of $\hat{D}$, then the abstract random variable in $\mathcal{H}$ corresponding to the short rate r_t is given by $\hat{r} = -2\hat{D}_{\rm s}$.

If we further define the mean-adjusted operator

$$\tilde{D} = \hat{D} - \frac{\langle \xi_t | \hat{D} | \xi_t \rangle}{\langle \xi_t | \xi_t \rangle}\, \hat{\mathbb{1}}, \tag{25}$$

then we deduce for the dynamical equation of the term structure state vector:

$$\mathrm{d}|\xi_t\rangle = \left(\tilde{D} - \tfrac{1}{8}\hat{\sigma}_t^2\right)|\xi_t\rangle \mathrm{d}t + \tfrac{1}{2}\hat{\sigma}_t|\xi_t\rangle \mathrm{d}W_t, \tag{26}$$

where

$$\hat{\sigma}_t = \hat{\nu}_t - \frac{\langle \xi_t | \hat{\nu}_t | \xi_t \rangle}{\langle \xi_t | \xi_t \rangle}\, \hat{\mathbb{1}}. \tag{27}$$

Projectively, the first term in the drift is a negative ‘Hamiltonian’ gradient flow $-\nabla r$; whereas the second term in the drift ensures the martingale property.

Information and interest

We now consider how interest rate dynamics is generated from elementary economic considerations.

There are of course numerous economic factors that affect the movement of interest rates, and causal relations that hold between these factors are often difficult to disentangle.

Hence, rather than attempting to address a range of factors simultaneously, we will focus on one key factor that appears important.

This is the liquidity risk, in the narrow sense of cash demand.

Our objective is to build an information-based model that reflects the market perception of future liquidity risk, and use it for the pricing and general risk management of interest rate derivatives.

To illustrate the role played by liquidity risk in determining interest rate systems, let us begin by examining deterministic term structures.

We remark that it is reasonable to regard the random variable X as representing the occurrence time of future liquidity issues, at least to first approximation.

From the viewpoint of the buyer of a bond, if with high probability cash is needed before the maturity T, then purchase will be made only if the bond price is sufficiently low.

Likewise, the seller of a bond would be willing to pay a high premium if there is a likely need for cash before time T.

Thus P_{0T} represents a survival function, where 'survival' means lack of liquidity crisis.

In fact, the price of a discount bond with maturity T is determined by the risk-adjusted probability that the liquidity crisis arises beyond time T:

$$P_{0T} = \mathbb{Q}(X \geq T). \tag{28}$$

The risk-neutral hazard rate associated with liquidity crisis is just the initial forward rate f_{0T}.

Therefore, for a small $\mathrm{d}T$ we have

$$f_{0T}\,\mathrm{d}T = \mathbb{Q}\left(X \in [T, T + \mathrm{d}T] \,|\, X \geq T\right). \tag{29}$$

That is, $f_{0T}\,\mathrm{d}T$ is the *a priori* risk-neutral probability of a liquidity crisis occurring in an infinitesimal interval $[T, T+\mathrm{d}T]$, conditional upon survival until time T.

More generally, in the case of a deterministic interest rate system, we have:

$$P_{tT} = \mathbb{Q}\,(X \geq T \,|\, X \geq t)\,. \tag{30}$$

Market information about future liquidity

Our aim now is to extend the deterministic model (30) into a dynamical one without losing the key economic interpretation.

The problem therefore is to identify the relevant conditioning.

In a dynamical setup, market participants accumulate noisy information concerning future liquidity risk.

It is this noisy observation of the timing X of the future cash demand that generates random movements in the bond price.

Thus if we let $\{\mathcal{F}_t\}$ denote the information generated by this observation, then the price of a discount bond is given by the conditional probability

$$P_{tT} = \mathbb{Q}\left(X \geq T \,|\, (X \geq t) \cap \mathcal{F}_t\right). \tag{31}$$

If we apply the Bayes formula, then (31) can be expressed in the form

$$P_{tT} = \frac{\mathbb{Q}\left(X \geq T | \mathcal{F}_t\right)}{\mathbb{Q}\left(X \geq t | \mathcal{F}_t\right)}. \tag{32}$$

This is the pricing formula for a discount bond that we propose here.

An elementary model for information and bond price

Let us now consider the problem of introducing a specific model for $\{\mathcal{F}_t\}$.

Since in the present formulation what concerns market participants is the value of X, the 'signal' component of the observation must be generated in some form by X itself.

In addition, there is an independent noise that obscures the value of X.

We consider a simple model whereby the information concerning the value of X

is revealed to the market linearly in time at a constant rate σ, and the noise is generated by an independent $\mathbb{Q}$-Brownian motion $\{B_t\}$.

Thus the information generating process is given by

$$\xi_t = \sigma t \phi(X) + B_t, \tag{33}$$

where $\phi(x)$ is a smooth invertible function.

In other words, we assume that the filtration $\mathcal{F}_t$ is given by the sigma algebra generated by $\{\xi_s\}_{0\leq s\leq t}$.

As regards the choice of the function $\phi(x)$ we shall have more to say shortly, but let us for the moment proceed with generality.

We note that since the magnitude of the signal-to-noise ratio is given by $\sigma\sqrt{t}$, the value of X will be revealed asymptotically, that is, X is $\mathcal{F}_\infty$-measurable.

Along with the fact that $\{\xi_t\}$ of (33) is Markovian, we find that the bond pricing formula simplifies in this model to

$$P_{tT} = \frac{\mathbb{Q}\left(X \geq T | \xi_t\right)}{\mathbb{Q}\left(X \geq t | \xi_t\right)}. \tag{34}$$

A calculation gives the following expression:

$$P_{tT} = \frac{\int_T^\infty \rho_0(x)\, \mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t}\mathrm{d}x}{\int_t^\infty \rho_0(x)\, \mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t}\mathrm{d}x}. \tag{35}$$

The model can be calibrated exactly against the initial yield curve according to the prescription $\rho_0(x) = -\partial_x P_{0x}$.

The subsequent evolution is then determined by the Markovian market information process.

The remaining degree of freedom, namely, the parameter σ, can be calibrated by use of derivative prices.

From the bond price (35) we can infer the implied short rate $r_t = -\partial_T P_{tT}|_{T=t}$.

This is given by

$$r_t = \frac{\rho_0(t)\, \mathrm{e}^{\sigma\phi(t)\xi_t - \frac{1}{2}\sigma^2\phi^2(t)t}}{\int_t^\infty \rho_0(x)\, \mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t}\mathrm{d}x}. \tag{36}$$

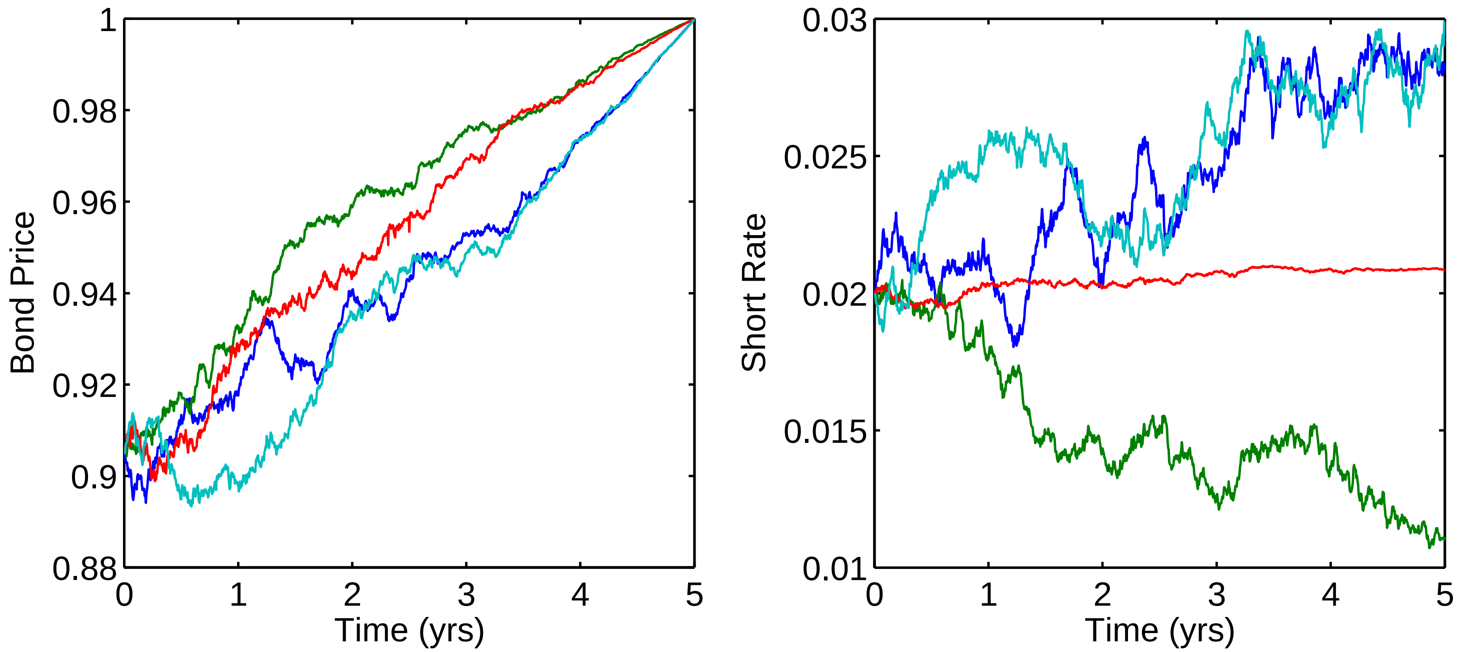

Figure 5: Sample paths of the discount function (35) and the associated short rate (36). The information-adjusting function is set as $\phi(x) = \mathrm{e}^{-0.025x}$, and the initial term structure is assumed flat so that $P_{0T} = \mathrm{e}^{-0.02T}$. The information flow rate is set as $\sigma = 0.3$, and the bond maturity is 5 years.

The instantaneous forward rate $f_{tT} = -\partial_T \ln P_{tT}$ is expressed analogously as

$$f_{tT} = \frac{\rho_0(T)\,\mathrm{e}^{\sigma\phi(T)\xi_t - \frac{1}{2}\sigma^2\phi^2(T)t}}{\int_T^\infty \rho_0(x)\,\mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t}\mathrm{d}x}. \tag{37}$$

Example. Consider a flat initial term structure given by $P_{0T} = \mathrm{e}^{-rT}$. The associated *a priori* density function is then exponential: $\rho_0(T) = r\mathrm{e}^{-rT}$.

In a *linear information model*, we have $\phi(x) = x$. Substitution of these in (35) yields the following bond price process

$$P_{tT} = \frac{N\left(\frac{\xi_t - r/\sigma}{\sqrt{t}} - \sigma T\sqrt{t}\right)}{N\left(\frac{\xi_t - r/\sigma}{\sqrt{t}} - \sigma t\sqrt{t}\right)}, \tag{38}$$

where $N(x)$ is the normal distribution function.

We remark that the interest rate dynamics can be driven by a other Lévy processes.

An example is given by the gamma filter, whereby the information process is

$$\xi_t = X\gamma_t. \tag{39}$$

Here $\{\gamma_t\}$ denotes a standard gamma process with rate parameter m.

In this case, the expression for the bond price process reads

$$P_{tT} = \frac{\int_T^\infty \rho_0(x) x^{-mt} \mathrm{e}^{-\xi_t/x} \mathrm{d}x}{\int_t^\infty \rho_0(x) x^{-mt} \mathrm{e}^{-\xi_t/x} \mathrm{d}x}, \tag{40}$$

and the associated short rate process is

$$r_t = \frac{\rho_0(t) t^{-mt} \mathrm{e}^{-\xi_t/t}}{\int_t^\infty \rho_0(x) x^{-mt} \mathrm{e}^{-\xi_t/x} \mathrm{d}x}. \tag{41}$$

Hence the present framework provides for a wide range of new interest rate models to be created that are tractable and relatively easy to implement.

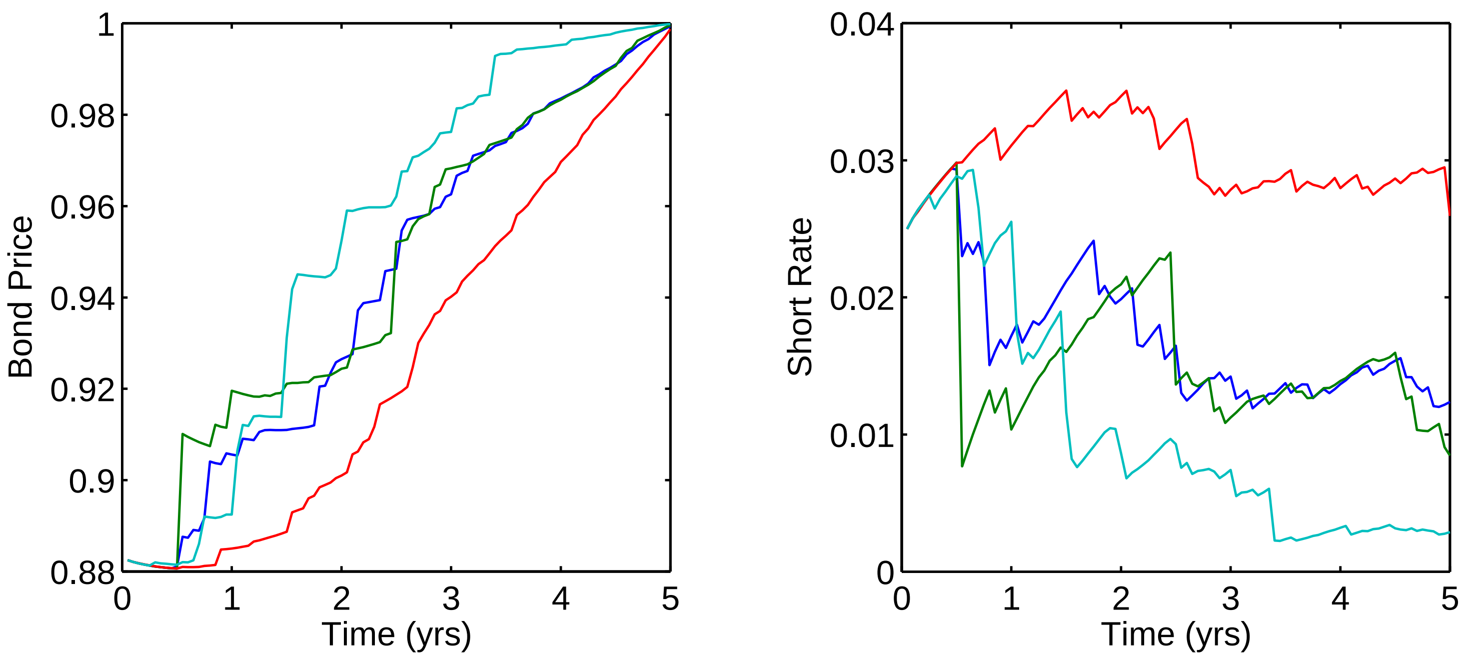

Figure 6: Sample paths of the discount function (40) and the associated short rate (41). The information-adjusting function is set as $\phi(x) = \mathrm{e}^{-0.02x}$, and the initial term structure is assumed flat so that $P_{0T} = \mathrm{e}^{-0.025T}$. The rate is set as $m = 0.1$, and the bond maturity is 5 years.

Information and geometry

We see that the function $\phi(x)$ appearing in the information process

$$\xi_t = \sigma t \phi(X) + B_t \tag{42}$$

embodies a significant economic impact.

It is therefore natural to enquire how much separation are there in a pair of economies characterised by different information flows:

$$\xi_t = \sigma t \phi(X) + B_t \quad \text{and} \quad \eta_t = \sigma t \psi(X) + B_t. \tag{43}$$

Since term structure density process takes the form

$$\rho_t^{\xi}(x) = \frac{\rho_0(x)\, \mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t}}{\int_t^{\infty} \rho_0(x)\, \mathrm{e}^{\sigma\phi(x)\xi_t - \frac{1}{2}\sigma^2\phi^2(x)t} \mathrm{d}x}, \tag{44}$$

we can compute the separation via

$$D(\phi,\psi) = \cos^{-1} \int_t^{\infty} \sqrt{\rho_t^{\xi}(x)\rho_t^{\eta}(x)} \mathrm{d}x. \tag{45}$$

Discussion: liquidity effect vs Fisher effect

Empirical studies indicate that a persistent increase in money supply leads in short term (up to a month or so) to a fall in nominal interest rates.

This is the so-called liquidity effect.

On the other hand, in the longer term an increase in money supply increases expected inflation, hence leading to an increase in nominal rates.

This is the so-called Fisher effect.

Typically both effects coexist in that an increase in money supply reduces nominal rates but increases expected inflation so that the real rate also falls.

Needless to say, interrelations between these effects are difficult to disentangle.

The implication of these macroeconomic considerations to the present approach is that the random variable X, which we identified as representing the timing of liquidity crisis in the narrow sense of cash demand, is dependant on a number of market factors and not merely on money supply.

Our objective here has been the introduction of a new interest rate modelling framework that captures some important macroeconomic elements, in such a way that resulting models can be used in practice for the pricing and risk management of interest rate derivatives.

The random variable X, whose existence is ensured by the positivity of nominal rates and the vanishing of infinite-maturity bond prices, has the dimension of time, and hence it has been interpreted as representing the timing of future liquidity crises.

It is worth remarking that the method of information geometry has not been fully exploited in the context of interest rate modelling.

There is a potential for significant further research into this area, with a range of practical applications.