Measures of quantum information

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Skew information

Wigner and Yanase proposed to find a measure of our knowledge of a difficult-to-measure observable with respect to a conserved quantity.

They discussed a number of postulates that such a measure should satisfy and proposed, tentatively, the so called *skew information* defined by

$$I_\rho(A) = -\frac{1}{2}\mathrm{Tr}\,[\rho^{\frac{1}{2}}, A]^2,$$

where ρ is a state (density matrix) and A is an observable

The postulates included the requirement from thermodynamics that knowledge decreases under the mixing of states; or put equivalently, that the proposed measure is a convex function in the state ρ.

The measure should also be additive with respect to the aggregation of isolated subsystems and, for an isolated system, independent of time.

The Dyson generalization of skew information

Dyson proposed that other measures, in particular the expression

$$I_\rho(p, A) = -\frac{1}{2} \operatorname{Tr}[\rho^p, A][\rho^{1-p}, A] \qquad 0 < p < 1,$$

may have the same general properties as the skew information.

Convexity of this expression in ρ became the celebrated Wigner-Yanase-Dyson conjecture which was later proved by Lieb.

The superadditivity conjecture

In the process that is the opposite of mixing, the information content should decrease.

This requirement comes from thermodynamics where it is satisfied for both classical and quantum mechanical systems. It reflects the loss of information about statistical correlations between two subsystems when they are only considered separately.

Wigner and Yanase conjectured that the skew information also possesses this property. They proved it when the state of the aggregated system is pure. We subsequently demonstrated that the conjecture fails for general mixed states.

Quantum statistics

Chentsov and Morozova defined a monotone metric as a map $\rho \to K_\rho$ from density matrices to sesquilinear forms on $M_n(\mathbf{C})$ satisfying:

(i) $K_\rho(A, A) \geq 0, \quad (= 0 \Leftrightarrow A = 0)$.

(ii) $K_\rho(A, B) = K_\rho(B^*, A^*)$

(iii) $\rho \to K_\rho(A, A)$ is continuous on $\mathcal{M}_n$.

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(iii) $\rho \to K_\rho(A, A)$ is continuous on $\mathcal{M}_n$.

(iv) $K_{T(\rho)}(T(A), T(A)) \leq K_\rho(A, A)$ for stochastic mappings

$$T : M_n(\mathbf{C}) \to M_m(\mathbf{C}).$$

A stochastic map is a linear trace preserving completely positive map.

Chentsov-Morozova's theorem

Consider a basis in which ρ with eigenvalues $\lambda_1, \ldots, \lambda_n$ is diagonalized. Then K_ρ is of the form

$$K_\rho(A, A) = C \sum_{i=1}^{n} \lambda_i^{-1} |A_{ii}|^2 + \sum_{i \neq j} |A_{ij}|^2 c(\lambda_i, \lambda_j),$$

where the function $c \colon \mathbf{R}_+^2 \to \mathbf{R}$ is symmetric and

$$c(x, x) = Cx^{-1}, \quad c(tx, ty) = t^{-1} c(x, y).$$

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Theorem (Petz)

A monotone metric is defined if and only if c is of the form

$$c(x, y) = \frac{1}{y f(xy^{-1})} \qquad f : \mathbf{R}_+ \to \mathbf{R}_+$$

where f is operator monotone, $f(t) = t f(t^{-1})$ for $t > 0$, and $f(1) = 1$.

The set $\mathcal{F}_{\text{op}}$

$\mathcal{F}_{\text{op}}$ are the functions $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

(i) f is operator monotone,

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Theorem

A function $f \in \mathcal{F}_{op}$ admits a canonical representation

$$f(t) = \frac{1+t}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(\lambda + t)(1 + \lambda t)(1 + \lambda)^2}\, h(\lambda)\, d\lambda$$

where $h : [0,1] \to [0,1]$ is measurable and (up to equivalence) uniquely determined by f.

Metric adjusted skew information

A Morozova-Chentsov function c is said to be regular, if the metric constant

$$m(c) = \lim_{t \to 0} c(t,1)^{-1} = f(0) \qquad f \in \mathcal{F}_{\rm op}$$

is strictly positive. We say also that the corresponding operator monotone function f is regular.

The metric adjusted skew information $I_\rho^c(A)$ is defined by setting

$$I_\rho^c(A) = \frac{m(c)}{2} K_\rho^c(i[\rho, A], i[\rho, A]).$$

where

$$c(x,y) = \frac{1}{y f(x y^{-1})} \qquad x, y > 0$$

for a regular function $f \in \mathcal{F}_{\rm op}$.

The representation of the WYD-information

Consider the weight functions

$$h_p(\lambda) = \frac{1}{\pi} \arctan \frac{(\lambda^p + \lambda^{1-p}) \sin p\pi}{1 - \lambda - (\lambda^p - \lambda^{1-p}) \cos p\pi} \qquad 0 < \lambda < 1$$

for $0 < p < 1$. The functions

$$\begin{aligned} f_p(t) &= \frac{1+t}{2} \exp \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(\lambda + t)(1 + \lambda t)(1 + \lambda)^2} h_p(\lambda)\, d\lambda \\ &= p(1-p) \cdot \frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)} \qquad t > 0 \end{aligned}$$

are regular in $\mathcal{F}_{\text{op}}$. The corresponding metric adjusted skew information

$$I_\rho^{c_p}(A) = -\frac{1}{2} \operatorname{Tr}[\rho^p, A][\rho^{1-p}, A],$$

where $c_p(x, y) = 1/(y f_p(xy^{-1}))$ for $x, y > 0$.

Affine representation of Morozova-Chentson functions

A Morozova-Chentsov function c allows a canonical representation

$$c(x,y) = \int_0^1 c_\lambda(x,y)\, d\mu_c(\lambda) \qquad x,y > 0,$$

where μ_c is a finite Borel measure on $[0,1]$ and

$$c_\lambda(x,y) = \frac{1+\lambda}{2}\left(\frac{1}{x+\lambda y} + \frac{1}{\lambda x + y}\right) \qquad \lambda \in [0,1]. \tag{1}$$

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It is regular if and only if

$$\left(\frac{1}{m(c)} =\right) \qquad \int_0^1 \frac{(1+\lambda)^2}{2\lambda}\, d\mu_c(\lambda) < \infty.$$

Alternative representation

The metric adjusted skew information (with A self-adjoint) may be written on the form

$$I_\rho^c(A) = \operatorname{Tr} \rho A^2 - \frac{m(c)}{2} \operatorname{Tr} A\, d_c(L_\rho, R_\rho) A,$$

where

$$d_c(x,y) = \int_0^1 xy \cdot c_\lambda(x,y) \frac{(1+\lambda)^2}{\lambda}\, d\mu_c(\lambda)$$

is operator concave in $[0,\infty) \times [0,\infty) \setminus (0,0)$.

The metric adjusted skew information may therefore be extended from the state manifold to the state space.

General results

(i) $\rho \to I^c_\rho(A)$ is a convex function.

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(iv) For any pure state ρ we have $I^c_\rho(A) = \mathrm{Var}_\rho(A)$.

(v) For any density matrix ρ we have

$$0 \le I^c_\rho(A) \le \mathrm{Var}_\rho(A).$$

$\mathcal{F}_{\text{op}}$ is a lattice with the order relation $\preceq$

Let $f, g \in \mathcal{F}_{\text{op}}$ and consider the function

$$\varphi(t) = \frac{t+1}{2} \frac{f(t)}{g(t)} \qquad t > 0.$$

We write $f \preceq g$ if $\varphi \in \mathcal{F}_{\text{op}}$. This definition renders $\mathcal{F}_{\text{op}}$ into a lattice with

$$f_{\min}(t) = \frac{2t}{t+1} \quad \text{and} \quad f_{\max}(t) = \frac{1+t}{2}$$

as minimal respectively maximal element.

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Theorem

$f \preceq g$ if and only if the representing functions satisfy $h_f \geq h_g$ a.e.

Optimality of the Wigner-Yanase skew information

The functions

$$f_p(t) = p(1-p) \cdot \frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)} \qquad 0 < p < 1$$

generate the Wigner-Yanase-Dyson skew-informations, and they are represented by the weight functions

$$h_p(\lambda) = \frac{1}{\pi} \arctan \frac{(\lambda^p + \lambda^{1-p}) \sin p\pi}{1 - \lambda - (\lambda^p - \lambda^{1-p}) \cos p\pi}$$

in the exponential representation theorem for the elements in $\mathcal{F}_{\mathrm{op}}$.

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Theorem

$f_p \preceq f_q$ for $p \leq q \leq 1/2$.

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Theorem

$f_p \preceq f_q$ for $p \leq q \leq 1/2$.

The statement is equivalent to operator monotonicity of the function

$$t \to \frac{(t+1)(t^q - 1)(t^{1-q} - 1)}{(t^p - 1)(t^{1-p} - 1)}\,.$$

Proof of optimality I

We consider a fixed $\lambda \in (0,1)$, set $z_0 = -\lambda + i\varepsilon$ for a small $\varepsilon > 0$, and obtain

$$h_p(\lambda) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} f_\varepsilon(p),$$

where

$$\begin{aligned} f_\varepsilon(p) &= \arg((1 - z_0^p)(1 - z_0^{1-p})) \\ &= \arg(1 - z_0^p) + \arg(1 - z_0^{1-p}). \end{aligned}$$

Since $f_\varepsilon(p) = f_\varepsilon(1-p)$ it suffices to show that $\arg(1 - z_0^p)$ is concave in p over $[0,1]$.

Proof of optimality II

We intend to show that the function

$$g(p) = \arg(1 - z^p) \qquad p \in [0, 1]$$

is concave for any z in

$$\mathcal{I} = \{z \mid \Im z > 0,\ |z| < 1\}$$

including z_0 for sufficiently small $\varepsilon > 0$. Since $\arg = \Im \log$, the second derivative

$$g''(p) = \Im \frac{-z^p(\log z)^2}{(1 - z^p)^2} = \frac{1}{p^2} \Im \frac{-z^p(\log z^p)^2}{(1 - z^p)^2} .$$

Proof of optimality III

We have to show that $g''(p)$ non-positive for $z \in \mathcal{I}$ and $0 \leq p \leq 1$.

In fact, it is enough to do this for $p = 1$ only, since for $z \in \mathcal{I}$ and $0 \leq p \leq 1$, then $z^p \in \mathcal{I}$ too.

The imaginary part of a complex number is non-positive if and only its complex argument is between π and 2π. Thus we need to show

$$q(z) = \arg \frac{-z(\log z)^2}{(1-z)^2} \in [\pi, 2\pi]$$

for every $z \in \mathcal{I}$.

Proof of optimality IV

Set $z = r\exp(i\theta)$, $0 < r < 1$, $0 \le \theta \le \pi$, then

$$\begin{aligned} q(z) &= \arg(-z) + 2\arg\log z - 2\arg(1-z) \\ &= \pi + \theta + 2\arctan\frac{\theta}{\log r} + 2\arctan\frac{r\sin\theta}{1 - r\cos\theta}\,. \end{aligned}$$

For $\theta = 0$, $q(z)$ is obviously π. For $\theta = \pi$,

$$q(z) = 2\pi + 2\arctan\frac{\pi}{\log r} < 2\pi.$$

We will show that $q(z)$ increases with θ for a fixed r, thus q is between π and 2π for $z \in \mathcal{I}$.

Proof of optimality V

The first derivative

$$\frac{\partial q}{\partial \theta} = 2\frac{\log r}{(\log r)^2 + \theta^2} + \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} .$$

Because of the inequality $\cos\theta \geq 1 - \theta^2/2$, we obtain a lower bound

$$\begin{aligned} \frac{\partial q}{\partial \theta} &\geq \frac{1 - r^2}{(1 - r)^2 + r\theta^2} + 2\frac{\log r}{(\log r)^2 + \theta^2} \\ &= \frac{\phi(r)\log r + \theta^2\psi(r)}{((1 - r)^2 + r\theta^2)((\log r)^2 + \theta^2)} \geq 0, \end{aligned}$$

where $\phi(r) = (1 - r^2)\log r + 2(1 - r)^2$ and $\psi(r) = 1 - r^2 + 2r\log r$.

The statement follows since $\phi \leq 0$ and $\psi \geq 0$. **QED**

The partial trace

Let ρ_{12} be an operator on a tensor product

$$H_{12} = H_1 \otimes H_2$$

of two Hilbert spaces.

Definition

The partial trace $\rho_1 = \mathrm{Tr}_2\, \rho_{12}$ is the operator on H_1 defined by setting

$$(\xi \mid \rho_1 \eta) = \sum_{i \in I} (\xi \otimes e_i \mid \rho_{12}(\eta \otimes e_i)) \quad \xi, \eta \in H_1$$

where $(e_i)_{i \in I}$ is any orthonormal basis in H_2.

Wigner and Yanase's superadditivity conjecture

Let k_1 (k_2) be self-adjoint operators on H_1 (H_2) and define

$$k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2.$$

Wigner and Yanase conjectured (1963) that

$$I_{\rho_{12}}(k_{12}) \geq I_{\rho_1}(k_1) + I_{\rho_2}(k_2),$$

where ρ_{12} is a bipartite state on $H_1 \otimes H_2$ and ρ_1 and ρ_2 are the partial traces of ρ_{12} on the first and second party.

They proved the conjecture when the state ρ_{12} is pure.

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Theorem (2006)

The skew information is not superadditive.

One-party superadditivity

Lemma

Let ρ be a bipartite density operator on a tensor product $H_1 \otimes H_2$ of two parties. If A is an observable of the first party then

$$I_\rho^c(A \otimes \mathbf{1}_2) \geq I_{\rho_1}^c(A),$$

where $\rho_1 = \mathrm{Tr}_2\, \rho$ is the partial trace of ρ on H_1.

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where $\rho_1 = \operatorname{Tr}_2 \rho$ is the partial trace of ρ on H_1.

Since the partial trace is completely positive and trace preserving and the metric K^c_ρ is decreasing we obtain

$$\begin{aligned} I^c_\rho(A \otimes \mathbf{1}_2) &= \frac{m(c)}{2} K^c_\rho(i[\rho, A \otimes \mathbf{1}_2], i[\rho, A \otimes \mathbf{1}_2]) \\ &\geq \frac{m(c)}{2} K^c_{\rho_1}(\operatorname{Tr}_2 i[\rho, A \otimes \mathbf{1}_2], \operatorname{Tr}_2 i[\rho, A \otimes \mathbf{1}_2]) \\ &= \frac{m(c)}{2} K^c_{\rho_1}(i[\rho_1, A], i[\rho_1, A]) = I^c_{\rho_1}(A). \end{aligned}$$

Semi-quantum states

A local von Neumann measurement P of the first party of a bipartite state ρ on a tensor product $H_1 \otimes H_2$ is given by

$$P(\rho) = \sum_{i \in I} (P_i \otimes \mathbf{1}_2)\rho(P_i \otimes \mathbf{1}_2)$$

where $\{P_i\}_{i \in I}$ is a resolution of the identity on H_1.

Definition

A bipartite state ρ is called a semi-quantum state if there exists a local von Neumann measurement $P = \{P_i\}_{i \in I}$ of the first (or second) party leaving ρ invariant, i.e. $P(\rho) = \rho$.

A state is semi-quantum if and only if

$$\rho = \sum_{i \in I} p_i P_i \otimes \rho_i$$

where $(p_i)_{i \in I}$ is a probability distribution and ρ_i for each $i \in I$ is a state of the second party.

Superadditivity for semi-quantum states

Theorem

Let ρ be a semi-quantum state on a tensor product $H_1 \otimes H_2$ of two parties, then we obtain superadditivity of the metric adjusted skew information

$$I_\rho^c(A \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes B) \geq I_{\rho_1}^c(A) + I_{\rho_2}^c(B).$$

The Effros tranform

Theorem

Suppose that h is operator convex. When restricted to positive commuting matrices the function g defined by

$$g(L, R) = h\left(\frac{L}{R}\right) R \tag{2}$$

is jointly convex, that is if the commutator $[L, R] = 0$ *and*

$$L = \lambda L_1 + (1 - \lambda) L_2 \quad \textit{and} \quad R = \lambda R_1 + (1 - \lambda) R_2$$

where also the commutators $[L_1, R_1] = 0$ *and* $[L_2, R_2] = 0$, *then*

$$g(L, R) \leq \lambda g(L_1, R_1) + (1 - \lambda) g(L_2, R_2)$$

for $0 \leq \lambda \leq 1$.

New proof of convexity

The metric adjusted skew information may be written on the form

$$I^c_\rho(A) = \frac{m(c)}{2} \operatorname{Tr} A\, \hat{c}(L_\rho, R_\rho)A,$$

where

$$\hat{c}(x,y) = (x-y)^2 c(x,y) \qquad x, y > 0. \tag{3}$$

The metric adjusted skew information is convex in the state variable ρ if $\hat{c}$ is operator convex, but $\hat{c}$ is the Effros transform of the function

$$h(t) = \frac{(t-1)^2}{f(t)} \qquad t > 0.$$

The statement therefore follows if h is operator convex.

Some operator convex functions

Theorem

Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be an operator monotone function. Then the function

$$h(t) = \frac{(t-1)^2}{f(t)} \qquad t > 0$$

is operator convex.

Notice that the theorem cannot be inverted. There are operator convex functions $h \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that the function $f(t) = (t-1)^2/h(t)$ is not operator monotone.

Definition

We introduce the (unbounded) metric adjusted skew information associated with a non-regular monotone metric by setting

$$I^c_\rho(A) = K^c_\rho(i[\rho, A], i[\rho, A])$$

if c is a non-regular Morozova-Chentsov function.

This type of metric adjusted skew information is unbounded and can no longer be extended from the state manifold to the state space. However, it enjoys all the same general properties as a bounded metric adjusted skew information.

A family of non-regular metrics I

Theorem

The functions

$$f_p(t) = p(1-p) \cdot \frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)}$$

are for $1 < p \leq 2$ *non-regular functions in* $\mathcal{F}_{op}$.

We only need to prove operator monotonicity and consider the identity

$$f_p(t) = -p(1-p)\frac{t-1}{g_p(t) - 1}$$

where the function

$$g_p(t) = \begin{cases} \dfrac{t^p - 1}{t - 1} + \dfrac{t^{1-p} - 1}{t - 1} & t > 0,\ t \neq 1 \\ 1 & t = 1. \end{cases}$$

A family of non-regular metrics II

Since both t^p and t^{1-p} are operator convex for $1 < p \leq 2$, it follows that g_p is operator monotone, and it is therefore also operator concave (notice that this conclusion does not require g_p to be positive).

By appealing to Bendat and Sherman's theorem once more and taking inverse we conclude that f_p is operator monotone.

Unbounded extension of the WYD-information

For $1 < p \leq 2$ the (unbounded) metric adjusted skew information associated with the non-regular functions $f_p \in \mathcal{F}_{op}$ is given by

$$I^c_\rho(A) = \frac{-1}{p(1-p)} \operatorname{Tr}[\rho^p, A] \cdot [\rho^{1-p}, A].$$

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In particular, this extension of the WYD-information is also associated with monotone metrics and can be understood in terms of the notion of metric adjusted skew information. The main difference is that the metric is regular for $0 < p < 1$ but non-regular for $1 < p \leq 2$.

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$$I_\rho^c(A) = \frac{-1}{p(1-p)} \operatorname{Tr}[\rho^p, A] \cdot [\rho^{1-p}, A].$$

But this is exactly the extension of the Wigner-Yanase-Dyson skew information to parameter values $1 < p \leq 2$ studied by Jencǒva and Ruskai, and earlier by Hasegawa.

In particular, this extension of the WYD-information is also associated with monotone metrics and can be understood in terms of the notion of metric adjusted skew information. The main difference is that the metric is regular for $0 < p < 1$ but non-regular for $1 < p \leq 2$.

It is therefore immediate that the extension is non-negative and convex in the state variable. Furthermore, it satisfies all the restricted forms of monotonicity under partial traces that we studied.

Monotone metrics for $-1 \leq p \leq 2$

Hasegawa and Petz proved operator monotonicity of the functions f_p for $0 < p < 1$. We proved operator monotonicity for $1 < p \leq 2$ and by symmetry for $-1 < p < 0$. We also notice that

$$f_p(t) \to \frac{t-1}{\log t} \qquad \text{for} \quad p \to 0 \quad \text{or} \quad p \to 1,$$

and this is the function generating the Kubo metric. Similarly,

$$f_p(t) = \frac{2t}{t+1} \qquad \text{for} \quad p = -1 \quad \text{or} \quad p = 2,$$

and this is the function generating the minimal monotone metric.

Therefore, with these extensions, we obtain monotone metrics for $-1 \leq p \leq 2$.