Riemannian metrics on positive definite matrices related to means

(joint work with Dénes Petz)

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Plan

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Reference

F. Hiai and D. Petz, Riemannian metrics on positive definite matrices related to means, *Linear Algebra Appl.* **430 (2009), 3105–3130.**

0. Motivation and introduction

Question?

For $n \times n$ positive definite matrices A, B and $0 < t < 1$, the well-known convergences of Lie-Trotter type are:

- $\lim_{\alpha \to 0} \left((1-t)A^\alpha + tB^\alpha\right)^{1/\alpha} = \exp((1-t)\log A + t\log B)$,
- $\lim_{\alpha \to 0} \left(A^\alpha \#_\alpha B^\alpha\right)^{1/\alpha} = \exp((1-t)\log A + t\log B)$.

What is the Riemannian geometry behind?

Can we explain these convergences in terms of Riemannian geometry?

Notation

- $\mathbb{M}_n$ **(the** $n \times n$ **complex matrices) is a Hilbert space with respect to the Hilbert-Schmidt inner product** $\langle X, Y \rangle_{\mathrm{HS}} := \mathrm{Tr}\, X^* Y$.
- $\mathbb{H}_n$ **(the** $n \times n$ **Hermitian matrices) is a real subspace of** $\mathbb{M}_n$**,** $\cong$ **the Euclidean space** $\mathbb{R}^{n^2}$.
- $\mathbb{P}_n$ **(the** $n \times n$ **positive definite matrices) is an open subset of** $\mathbb{H}_n$**, a smooth differentiable manifold with** $T_D\mathbb{P}_n = \mathbb{H}_n$.
- $\mathcal{D}_n$ **(the** $n \times n$ **positive definite matrices of trace** 1**) is a smooth differentiable submanifold of** $\mathbb{P}_n$ **with** $T_D\mathcal{D}_n = \{H \in \mathbb{H}_n : \mathrm{Tr}\, H = 0\}$.

A Riemannian metric $K_D(H, K)$ **is a family of inner products on** $\mathbb{H}_n$ **depending smoothly on the foot point** $D \in \mathbb{P}_n$.

For $D \in \mathbb{P}_n$, write

$$\mathbf{L}_D X := DX \quad \textbf{and} \quad \mathbf{R}_D X := XD, \qquad X \in \mathbb{M}_n.$$

$\mathbf{L}_D$ **and** $\mathbf{R}_D$ **are commuting positive operators on** $(\mathbb{M}_n, \langle \cdot, \cdot \rangle_{\mathrm{HS}})$**.**

Statistical Riemannian metric [Mostow, Skovgaard, Ohara-Suda-Amari, Lawson-Lim, Moakher, Bhatia-Holbrook]

$$g_D(H, K) := \mathrm{Tr}\, D^{-1} H D^{-1} K = \langle H, (\mathbf{L}_D \mathbf{R}_D)^{-1} K \rangle_{\mathrm{HS}}$$

This is considered as a geometry on the Gaussian distributions p_D **with zero mean and covariance matrix** D**. The Boltzmann entropy is**

$$S(p_D) = \frac{1}{2} \log(\det D) + \textbf{const.}$$

and

$$g_D(H, K) = \frac{\partial^2}{\partial s \partial t} S(p_{D+sH+tK}) \Big|_{s=t=0} \qquad \textbf{(Hessian)}.$$

Congruence-invariant **For any invertible $X \in \mathbb{M}_n$,**

$$g_{XDX^*}(XHX^*, XKX^*) = g_D(H, K)$$

Geodesic curve

$$\gamma(t) = A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \quad (0 \le t \le 1)$$

The geodesic midpoint $\gamma(1/2)$ is the geometric mean $A \# B$ [Pusz-Woronowicz].

Geodesic distance

$$\delta(A, B) = \| \log(A^{-1/2}BA^{-1/2})\|_{\mathrm{HS}}$$

This is the so-called Thompson metric.

Monotone metrics [Petz]

$$K_{\beta(D)}(\beta(X), \beta(Y)) \le K_D(X, Y)$$

if $\beta : \mathbb{M}_n \to \mathbb{M}_m$ is completely positive and trace-preserving.

Theorem (Petz, 1996) **There is a one-to-one correspondence:**

$$\left\{\textbf{monotone metrics with } K_D(I, I) = \operatorname{Tr} D^{-1}\right\}$$

$$\updownarrow$$

$$\left\{\textbf{operator monotone functions } f : (0, \infty) \to (0, \infty) \textbf{ with } f(1) = 1\right\}$$

by

$$K_D^f(X, Y) = \langle X, (\mathbf{J}_D^f)^{-1} Y \rangle_{\mathrm{HS}}, \qquad \mathbf{J}_D^f := f(\mathbf{L}_D \mathbf{R}_D^{-1}) \mathbf{R}_D.$$

K_D^f is symmetric (i.e., $K_D^f(X, Y) = K_D^f(Y^*, X^*)$) if and only if f is symmetric, (i.e., $x f(x^{-1}) = f(x)$).

A symmetric monotone metric is also called a quantum Fisher information.

Theorem (Kubo-Ando, 1980) **There is a one-to-one correspondence:**

$$\{\textbf{operator means } \sigma\}$$

$$\updownarrow$$

$$\{\textbf{operator monotone functions } f : (0, \infty) \to (0, \infty) \textbf{ with } f(1) = 1\}$$

by

$$A\,\sigma_f\,B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \qquad A, B \in \mathbb{P}_n.$$

σ_f **is symmetric if and only if** f **is symmetric.**

Quantum skew information

When $0 < p < 1$, the Wigner-Yanase-Dyson skew information is

$$I_D^{\mathrm{WYD}}(p,K) := -\frac{1}{2}\mathrm{Tr}\,[D^p,K][D^{1-p},K] = \frac{p(1-p)}{2}K_D^{f_p}(i[D,K],i[D,K])$$

for $D \in \mathcal{D}_n$, $K \in \mathbb{H}_n$, where f_p is an operator monotone function:

$$f_p(x) := p(1-p)\,\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}.$$

For each operator monotone function f that is regular (i.e., $f(0) := \lim_{x\searrow 0} f(x) > 0$), Hansen introduced the metric adjusted skew information (or quantum skew information):

$$I_D^f(K) := \frac{f(0)}{2}\,K_D^f(i[D,K],i[D,K]), \qquad D \in \mathcal{D}_n,\ K \in \mathbb{H}_n.$$

Generalized covariance

For an operator monotone function f,

$$\varphi_D[H,K] := \langle H, \mathbf{J}_D^f K\rangle_{\mathrm{HS}}, \qquad D \in \mathcal{D}_n,\ \ H,K \in \mathbb{H}_n, \mathrm{Tr}\, H = \mathrm{Tr}\, K = 0,$$

$\varphi_D[H,K] = \mathrm{Tr}\, DHK$ **if** D **and** K **are commuting.**

Motivation **The above quantities are Riemannian metrics in the form**

$$K_D^{\phi}(H,K) := \langle H, \phi(\mathbf{L}_D, \mathbf{L}_R)^{-1} K\rangle_{\mathrm{HS}} = \sum_{i,j=1}^{k} \phi(\lambda_i, \lambda_j)^{-1} \mathrm{Tr}\, P_i H P_j K,$$

where $D = \sum_{i=1}^{k} \lambda_i P_i$ **is the spectral decomposition, and the kernel function** $\phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ **is in the form**

$$\phi(x,y) = M(x,y)^{\theta},$$

a degree $\theta \in \mathbb{R}$ **power of a certain mean** $M(x,y)$**. A systematic study is desirable, from the viewpoints of geodesic curves, scalar curvature, information geometry, etc.**

1. Geodesic shortest curve and geodesic distance

Let $\mathfrak{M}_0$ denote the set of smooth symmetric homogeneous means $M : (0,\infty)\times(0,\infty)\to(0,\infty)$ satisfying

- $M(x,y)=M(y,x)$**,**
- $M(\alpha x,\alpha y)=\alpha M(x,y)$**,** $\alpha>0$**,**
- $M(x,y)$ **is non-decreasing and smooth in** x,y**,**
- $\min\{x,y\}\le M(x,y)\le\max\{x,y\}$**.**

For $M\in\mathfrak{M}_0$ and $\theta\in\mathbb{R}$, define $\phi(x,y):=M(x,y)^\theta$ and consider a Riemannian metric on $\mathbb{P}_n$ given by

$$K_D^\phi(H,K):=\langle H,\phi(\mathbf{L}_D,\mathbf{R}_D)^{-1}K\rangle_{\mathrm{HS}},\qquad D\in\mathbb{P}_n,\ H,K\in\mathbb{H}_n.$$

When $D = U\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)U^*$ **is the diagonalization,**

$$\phi(\mathbf{L}_D, \mathbf{R}_D)^{-1/2}H = U\left(\left[\frac{1}{\sqrt{\phi(\lambda_i, \lambda_j)}}\right]_{ij} \circ (U^*HU)\right)U^*,$$

$$K_D^\phi(H, H) = \|\phi(\mathbf{L}_D, \mathbf{R}_D)^{-1/2}H\|_{\mathrm{HS}}^2 = \left\|\left[\frac{1}{\sqrt{\phi(\lambda_i, \lambda_j)}}\right]_{ij} \circ (U^*HU)\right\|_{\mathrm{HS}}^2,$$

where $\circ$ **denotes the Schur product.**

For a C^1 **curve** $\gamma : [0, 1] \to \mathbb{P}_n$**, the length of** γ **is**

$$L_\phi(\gamma) := \int_0^1 \sqrt{K_{\gamma(t)}^\phi(\gamma'(t), \gamma'(t))}\, dt = \int_0^1 \|\phi(\mathbf{L}_{\gamma(t)}, \mathbf{R}_{\gamma(t)})^{-1/2}\gamma'(t)\|_{\mathrm{HS}}\, dt.$$

The geodesic distance between A, B **is**

$$\delta_\phi(A, B) := \inf\left\{L_\phi(\gamma) : \gamma \textbf{ is a } C^1 \textbf{ curve joining } A, B\right\}.$$

A geodesic shortest curve is a γ **joining** A, B **s.t.** $L_\phi(\gamma) = \delta_\phi(A, B)$ **if exists.**

When $\phi(x,y) := M(x,y)^\theta$ for $M \in \mathfrak{M}_0$ and $\theta \in \mathbb{R}$ as above,

Theorem **Assume $A, B \in \mathbb{P}_n$ are commuting (i.e., $AB = BA$). Then, independently of the choice of $M \in \mathfrak{M}_0$, the following hold:**

- **The geodesic distance between A, B is**

$$\delta_\phi(A,B) = \begin{cases} \frac{2}{|2-\theta|} \left\| A^{\frac{2-\theta}{2}} - B^{\frac{2-\theta}{2}} \right\|_{\mathrm{HS}} & \textbf{if } \theta \neq 2, \\ \| \log A - \log B \|_{\mathrm{HS}} & \textbf{if } \theta = 2, \end{cases}$$

- **A geodesic shortest curve joining A, B is**

$$\gamma(t) := \begin{cases} \left((1-t)A^{\frac{2-\theta}{2}} + tB^{\frac{2-\theta}{2}} \right)^{\frac{2}{2-\theta}}, \quad 0 \le t \le 1 & \textbf{if } \theta \neq 2, \\ \exp((1-t)\log A + t \log B), \quad 0 \le t \le 1 & \textbf{if } \theta = 2, \end{cases}$$

- **If $M(x,y)$ is an operator monotone mean and $\theta = 1$, then $\gamma(t) = \left((1-t)A^{1/2} + tB^{1/2} \right)^2$, $0 \le t \le 1$, is a unique geodesic shortest curve joining A, B.**

Theorem $(\mathbb{P}_n, K^\phi)$ **is complete (i.e., the geodesic distance** $\delta_\phi(A,B)$ **is complete) if and only if** $\theta = 2$.

Proposition **For every** $M \in \mathfrak{M}_0$ **and** $A, B \in \mathbb{P}_n$ **there exists a smooth geodesic shortest curve for** K^ϕ **joining** A, B **whenever** θ **is sufficiently near** 2 **depending on** M **and** A, B.

2. Characterizing isometric transformation

For $N, M \in \mathfrak{M}_0$ **and** $\kappa, \theta \in \mathbb{R}$**, define** $\psi, \phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ **by**

$$\psi(x,y) := N(x,y)^{\kappa}, \qquad \phi(x,y) := M(x,y)^{\theta},$$

and Riemannian metrics K^{ψ}, K^{ϕ} **by**

$$K_D^{\psi}(H,K) := \langle H, \psi(\mathbf{L}_D, \mathbf{R}_D)^{-1} K \rangle_{\mathrm{HS}},$$
$$K_D^{\phi}(H,K) := \langle H, \phi(\mathbf{L}_D, \mathbf{R}_D)^{-1} K \rangle_{\mathrm{HS}}.$$

$F : (0,\infty) \to (0,\infty)$ **is an onto smooth function such that** $F'(x) \neq 0$ **for all** $x > 0$**.**

Theorem **When** $\alpha > 0$**, the transformation** $D \in \mathbb{P}_n \mapsto F(D) \in \mathbb{P}_n$ **is isometric from** $(\mathbb{P}_n, \alpha^2 K^{\phi})$ **onto** $(\mathbb{P}_n, K^{\psi})$ **if and only if one of the following** $(\mathbf{1}^{\circ})$**–**$(\mathbf{5}^{\circ})$ **holds:**

(1°) $\kappa = \theta = 0$ **and** $F(x) = \alpha x$**,** $x > 0$**. (**N, M **are irrelevant;** K^ψ **and** K^ϕ **are the Euclidean metric.)**

(2°) $\kappa = 0$**,** $\theta \neq 0, 2$ **and**

$$F(x) = \alpha\left|\frac{2}{2-\theta}\right| x^{\frac{2-\theta}{2}}, \qquad x > 0,$$

$$M(x,y) = \left(\frac{2-\theta}{2} \cdot \frac{x-y}{x^{\frac{2-\theta}{2}} - y^{\frac{2-\theta}{2}}}\right)^{2/\theta}, \qquad x, y > 0.$$

(N **is irrelevant;** K^ϕ **is a pull-back of the Euclidean metric.)**

(3°) $\kappa \neq 0, 2$**,** $\theta = 0$ **and**

$$F(x) = \left(\alpha\left|\frac{2-\kappa}{2}\right| x\right)^{\frac{2}{2-\kappa}}, \qquad x > 0,$$

$$N(x,y) = \left(\frac{2-\kappa}{2} \cdot \frac{x-y}{x^{\frac{2-\kappa}{2}} - y^{\frac{2-\kappa}{2}}}\right)^{2/\kappa}, \qquad x, y > 0.$$

(M **is irrelevant.)**

(4°) $\kappa, \theta \neq 0, 2$ **and**

$$F(x) = \left(\alpha\left|\frac{2-\kappa}{2-\theta}\right|\right)^{\frac{2}{2-\kappa}} x^{\frac{2-\theta}{2-\kappa}}, \qquad x > 0,$$

$$M(x,y) = \left(\frac{2-\theta}{2-\kappa} \cdot \frac{x-y}{x^{\frac{2-\theta}{2-\kappa}} - y^{\frac{2-\theta}{2-\kappa}}}\right)^{2/\theta} N\left(x^{\frac{2-\theta}{2-\kappa}}, y^{\frac{2-\theta}{2-\kappa}}\right)^{\kappa/\theta}, \qquad x, y > 0.$$

(5°) $\kappa = \theta = 2$ **and**

$$F(x) = cx^{\alpha}, \quad x > 0 \quad (c > 0 \textbf{ is a constant}),$$

$$M(x,y) = \alpha\left(\frac{x-y}{x^{\alpha} - y^{\alpha}}\right) N\left(x^{\alpha}, y^{\alpha}\right), \qquad x, y > 0,$$

or

$$F(x) = cx^{-\alpha}, \quad x > 0 \quad (c > 0 \textbf{ is a constant}),$$

$$M(x,y) = \alpha\left(\frac{x-y}{y^{-\alpha} - x^{-\alpha}}\right) N\left(x^{-\alpha}, y^{-\alpha}\right), \qquad x, y > 0.$$

3. Two kinds of isometric families of Riemannian metrics

For $N \in \mathfrak{M}_0$, $\kappa \in \mathbb{R} \setminus \{2\}$, $\theta \in \mathbb{R} \setminus \{0, 2\}$, and $\alpha \in \mathbb{R} \setminus \{0\}$, define

$$N_{\kappa,\theta}(x, y) := \left(\frac{2-\theta}{2-\kappa} \cdot \frac{x-y}{x^{\frac{2-\theta}{2-\kappa}} - y^{\frac{2-\theta}{2-\kappa}}} \right)^{2/\theta} N\left(x^{\frac{2-\theta}{2-\kappa}}, y^{\frac{2-\theta}{2-\kappa}} \right)^{\kappa/\theta},$$

$$N_\alpha(x, y) := \alpha \left(\frac{x-y}{x^\alpha - y^\alpha} \right) N\left(x^\alpha, y^\alpha\right), \qquad x, y > 0.$$

In particular, $N_{0,\theta}$'s are Stolarsky means

$$S_\theta(x, y) := \left(\frac{2-\theta}{2} \cdot \frac{x-y}{x^{\frac{2-\theta}{2}} - y^{\frac{2-\theta}{2}}} \right)^{2/\theta},$$

interpolating the following typical means:

$$S_{-2}(x,y) = M_{\mathrm{A}}(x,y) := \frac{x+y}{2} \quad (\textbf{arithmetic mean}),$$

$$S_1(x,y) = M_{\sqrt{\ }}(x,y) := \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \quad (\textbf{root mean}),$$

$$S_2(x,y) := \lim_{\theta\to 2} S_\theta(x,y) = M_{\mathrm{L}}(x,y) := \frac{x-y}{\log x - \log y} \quad (\textbf{logarithmic mean}),$$

$$S_4(x,y) = M_{\mathrm{G}}(x,y) := \sqrt{xy} \quad (\textbf{geometric mean}).$$

The metric corresponding to the root mean (called the Wigner-Yanase metric) is a unique monotone metric that is a pull-back of the Euclidean metric [Gibilisco-Isola].

Proposition

(a) For any $N \in \mathfrak{M}_0$**,** $\kappa \in \mathbb{R} \setminus \{2\}$**, and** $\theta \in \mathbb{R} \setminus \{0, 2\}$**,**

$$N_{\kappa,\theta}(x, y) = S_{\frac{2(\theta-\kappa)}{2-\kappa}}(x, y)^{\frac{2(\theta-\kappa)}{\theta(2-\kappa)}} N\left(x^{\frac{2-\theta}{2-\kappa}}, y^{\frac{2-\theta}{2-\kappa}}\right)^{\kappa/\theta},$$

$$\lim_{\theta \to 2} N_{\kappa,\theta}(x, y) = M_{\mathrm{L}}(x, y).$$

If $0 \le \kappa \le \theta < 2$ **or** $2 < \theta \le \kappa$**, then** $N_{\kappa,\theta} \in \mathfrak{M}_0$**.**

(b) For any $N \in \mathfrak{M}_0$ **and** $\alpha \in \mathbb{R} \setminus \{0\}$**,**

$$N_\alpha(x, y) = S_{2-2\alpha}(x, y)^{1-\alpha} N(x^\alpha, y^\alpha),$$

$$\lim_{\alpha \to 0} N_\alpha(x, y) = M_{\mathrm{L}}(x, y).$$

If $0 < \alpha \le 1$**, then** $N_\alpha \in \mathfrak{M}_0$**.**

For any $N \in \mathfrak{M}_0$, the above theorem and proposition show:

- **When $\kappa \geq 0$ and $\kappa \neq 2$,**
  $K^{N^\theta_{\kappa,\theta}}$ ($\kappa \leq \theta < 2$ **or** $\kappa \geq \theta > 2$) **is a one-parameter isometric family of Riemannian metrics starting from K^{N^κ} and converging to $K^{M^2_{\mathrm{L}}}$ as $\theta \to 2$.**

- **When $\kappa = 2$,**
  $K^{N^2_\alpha}$ $(1 \geq \alpha > 0)$ **is a one-parameter isometric family of Riemannian metrics starting from K^{N^2} and converging to $K^{M^2_{\mathrm{L}}}$ as $\alpha \to 0$.**

Claim **The metric $K^{M^2_{\mathrm{L}}}$ is an attractor among the Riemannian metrics K^{M^θ} $(M \in \mathfrak{M}_0,\ \theta \geq 0)$.**

The geodesic shortest curve for $K^{M^2_{\mathrm{L}}}$ joining $A, B \in \mathbb{P}_n$ is

$$\gamma_{A,B}(t) := \exp((1-t)\log A + t\log B) \qquad (0 \leq t \leq 1).$$

The geodesic distance between A, B with respect to $K^{M^2_{\mathrm{L}}}$ is

$$\delta_{M^2_{\mathrm{L}}}(A,B) := \|\log A - \log B\|_{\mathrm{HS}}.$$

Theorem **Let $N \in \mathfrak{M}_0$ and $A, B \in \mathbb{P}_n$ be arbitrary.**

(a) For the one-parameter family $K^{N^{\theta}_{\kappa,\theta}}$ $(0 \le \kappa \le \theta < 2$ or $\kappa \ge \theta > 2)$,

$$\delta_{N^{\theta}_{\kappa,\theta}}(A, B) = \delta_{N^{\kappa}}(A_{k,\theta}, B_{\kappa,\theta}) \longrightarrow \| \log A - \log B \|_{\mathrm{HS}} \quad (\theta \to 2),$$

where

$$A_{\kappa,\theta} := \left(\frac{2-\kappa}{2-\theta} \right)^{\frac{2}{2-\kappa}} A^{\frac{2-\theta}{2-\kappa}}, \qquad B_{\kappa,\theta} := \left(\frac{2-\kappa}{2-\theta} \right)^{\frac{2}{2-\kappa}} B^{\frac{2-\theta}{2-\kappa}}.$$

(b) For the one-parameter family $K^{N^{2}_{\alpha}}$ $(1 \ge \alpha > 0)$,

$$\delta_{N^2_\alpha}(A, B) = \frac{1}{\alpha}\,\delta_{N^2}(A^{\alpha}, B^{\alpha}) \longrightarrow \| \log A - \log B \|_{\mathrm{HS}} \quad (\alpha \searrow 0).$$

Theorem **Let $N \in \mathfrak{M}_0$ and $A, B \in \mathbb{P}_n$ be arbitrary. In the following, assume that geodesic shortest curves are always parametrized under constant speed.**

(a) If $\gamma_{A_{\kappa,\theta},B_{\kappa,\theta}}(t)$ is the geodesic shortest curve for K^{N^κ} joining $A_{\kappa,\theta}, B_{\kappa,\theta}$, then the geodesic shortest curve for $K^{N^\theta_{\kappa,\theta}}$ joining A, B is given by $\left(\frac{2-\theta}{2-\kappa}\right)^{\frac{2}{2-\theta}}\left(\gamma_{A_{\kappa,\theta},B_{\kappa,\theta}}(t)\right)^{\frac{2-\kappa}{2-\theta}}$ and

$$\lim_{\theta\to 2}\left(\frac{2-\theta}{2-\kappa}\right)^{\frac{2}{2-\theta}}\left(\gamma_{A_{\kappa,\theta},B_{\kappa,\theta}}(t)\right)^{\frac{2-\kappa}{2-\theta}} = \exp((1-t)\log A + t\log B) \quad (0 \le t \le 1).$$

(b) If $\gamma_{A^\alpha,B^\alpha}(t)$ is the geodesic shortest curve for K^{N^2} joining A^α, B^α, then the geodesic shortest curve for $K^{N^2_\alpha}$ joining A, B is given by $\left(\gamma_{A^\alpha,B^\alpha}(t)\right)^{1/\alpha}$ and

$$\lim_{\alpha\searrow 0}\left(\gamma_{A^\alpha,B^\alpha}(t)\right)^{1/\alpha} = \exp((1-t)\log A + t\log B) \quad (0 \le t \le 1).$$

The above convergences for the geodesic shortest curves may be considered as variations of the Lie-Trotter formula.

Examples

- **When $\kappa = 0$, $N_{0,\theta} = S_\theta$ is the family of Stolarsky means. The geodesic distance and the geodesic shortest curve for $K^{S_\theta^\theta}$ are**

$$\delta_{S_\theta^\theta}(A, B) = \frac{2}{|2-\theta|} \left\| A^{\frac{2-\theta}{2}} - B^{\frac{2-\theta}{2}} \right\|_{\mathrm{HS}},$$

$$\gamma_{A,B}(t) = \left((1-t)A^{\frac{2-\theta}{2}} + tB^{\frac{2-\theta}{2}} \right)^{\frac{2}{2-\theta}}.$$

We have

$$\lim_{\theta\to 2} \frac{2}{|2-\theta|} \left\| A^{\frac{2-\theta}{2}} - B^{\frac{2-\theta}{2}} \right\|_{\mathrm{HS}} = \| \log A - \log B \|_{\mathrm{HS}},$$

$$\lim_{\theta\to 2} \left((1-t)A^{\frac{2-\theta}{2}} + tB^{\frac{2-\theta}{2}} \right)^{\frac{2}{2-\theta}} = \exp((1-t)\log A + t\log B).$$

- **When** $N = M_{\mathrm{G}}$ **(geometric mean),** $K^{M_{\mathrm{G}}^2}$ **is the statistical Riemannian metric and** $N_\alpha(x,y) = \alpha\left(\dfrac{x-y}{x^\alpha - y^\alpha}\right)(xy)^{\alpha/2}$**,** $x, y > 0$**. The geodesic distance and the geodesic shortest curve for** $K^{N_\alpha^2}$ **are**

$$\delta_{N_\alpha^2}(A,B) = \frac{1}{\alpha}\,\delta_{M_{\mathrm{G}}^2}(A^\alpha, B^\alpha) = \big\| \log(A^{-\alpha/2}B^\alpha A^{-\alpha/2})^{1/\alpha}\big\|_{\mathrm{HS}},$$

$$\gamma_{A,B}(t) = (A^\alpha \#_t B^\alpha)^{1/\alpha}.$$

We have

$$\lim_{\alpha\to 0}\big\| \log(A^{-\alpha/2}B^\alpha A^{-\alpha/2})^{1/\alpha}\big\|_{\mathrm{HS}} = \|\log A - \log B\|_{\mathrm{HS}} \quad \textbf{(decreasing)},$$

$$\lim_{\alpha\to 0}(A^\alpha \#_t B^\alpha)^{1/\alpha} = \exp((1-t)\log A + t\log B).$$

Remark **When** σ **is an operator mean corresponding to an operator monotone function** f **and** $s := f'(1)$**,**

$$\lim_{\alpha\to 0}(A^\alpha \,\sigma\, B^\alpha)^{1/\alpha} = \exp((1-s)\log A + s\log B).$$

4. Comparison property

Theorem **Let** $\phi^{(1)}, \phi^{(2)} : (0,\infty)\times(0,\infty) \to (0,\infty)$ **be smooth symmetric kernel functions. The following conditions are equivalent:**

(i) $\phi^{(1)}(x,y) \le \phi^{(2)}(x,y)$ **for all** $x, y > 0$**;**

(ii) $K_D^{\phi^{(1)}}(H,H) \ge K_D^{\phi^{(2)}}(H,H)$ **for all** $D \in \mathbb{P}_n$ **and** $H \in \mathbb{H}_n$**;**

(iii) $L_{\phi^{(1)}}(\gamma) \ge L_{\phi^{(2)}}(\gamma)$ **for all** C^1 **curve** $\gamma \in \mathbb{P}_n$**;**

(iv) $\delta_{\phi^{(1)}}(A,B) \ge \delta_{\phi^{(2)}}(A,B)$ **for all** $A, B \in \mathbb{P}_n$**.**

For example, for $\theta \in \mathbb{R}$**, let** $\phi_\theta(x,y) := S_\theta(x,y)^\theta$ **and** $\phi(x,y) := M(x,y)^\theta$ **with** $M \in \mathfrak{M}_0$**. If** $\theta > 0$ **and** $M(x,y) \lesseqgtr S_\theta(x,y)$ **for all** $x, y > 0$**, then**

$$\delta_\phi(A,B) \gtreqless \delta_{\phi_\theta}(A,B) = \begin{cases} \frac{2}{|2-\theta|}\|A^{\frac{2-\theta}{2}} - B^{\frac{2-\theta}{2}}\|_{\mathrm{HS}} & \textbf{if } \theta \ne 2, \\ \|\log A - \log B\|_{\mathrm{HS}} & \textbf{if } \theta = 2. \end{cases}$$

Theorem **If $AB \neq BA$ and $\phi(x,y) \lesseqgtr \phi_\theta(x,y)$ for all $x, y > 0$ with $x \neq y$, then, $\delta_\phi(A,B) \gtreqless \delta_{\phi_\theta}(A,B)$.**

- **In the case $\theta = 2$ and $\phi(x,y) = M_{\mathrm{G}}(x,y)^2$,**

$$\|\log(A^{-1/2}BA^{-1/2})\|_{\mathrm{HS}} \geq \|\log A - \log B\|_{\mathrm{HS}}$$

(exponential metric increasing [Mostow, Bhatia, Bhatia-Holbrook])

- **In the case $\theta = 2$ and $\phi(x,y) = M_{\mathrm{A}}(x,y)^2$,**

$$\delta_{M_{\mathrm{A}}^2}(A,B) \leq \|\log A - \log B\|_{\mathrm{HS}}$$

(exponential metric decreasing)

- **In the case $\theta = 1$,**

$$\delta_{M_{\mathrm{G}}}(A,B) \geq \delta_{M_{\mathrm{L}}}(A,B) \geq 2\|A^{1/2} - B^{1/2}\|_{\mathrm{HS}} \geq \delta_{M_{\mathrm{A}}}(A,B)$$

Bogoliubov Wigner-Yanase Bures-Uhlmann

(square metric increasing/decreasing)

Unitarily invariant norms

For a unitarily invariant norm $|||\cdot|||$,

$$L_{\phi,|||\cdot|||}(\gamma) := \int_0^1 |||\phi(\mathbf{L}_{\gamma(t)}, \mathbf{R}_{\gamma(t)})^{-1/2}\gamma'(t)|||\,dt,$$

$$\delta_{\phi,|||\cdot|||}(A,B) := \inf\{L_{\phi,|||\cdot|||}(\gamma) : \gamma \text{ is a } C^1 \text{ curve joining } A, B\}.$$

$(\mathbb{P}_n, \delta_{\phi,|||\cdot|||})$ is no longer a Riemannian manifold but a differential manifold of Finsler type. Many results above hold true even when $\|\cdot\|_{\mathrm{HS}}$ is replaced by $|||\cdot|||$.

Let $\phi^{(k)}(x,y) := M^{(k)}(x,y)^\theta$, $k = 1, 2$. To compare $L_{\phi^{(1)},|||\cdot|||}(\gamma)$ and $L_{\phi^{(2)},|||\cdot|||}(\gamma)$, the infinite divisibility of $M^{(1)}(x,y)/M^{(2)}(x,y)$ is crucial:

$$\left(\frac{M^{(1)}(e^t,1)}{M^{(2)}(e^t,1)}\right)^r$$

is positive definite on $\mathbb{R}$ for any $r > 0$ [Bhatia-Kosaki, Kosaki].

5. Problems

- **Want to prove the unique existence of geodesic shortest curve between $A, B \in \mathbb{P}_n$ with respect to K^{ϕ}.**
- **Need to study $(\mathcal{D}_n, K^{\phi})$ rather than $(\mathbb{P}_n, K^{\phi})$ for applicatioins to quantum information.**

Thank you for your attention.