

Correlations in Quantum States

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Qualification and Quantification of

Correlations (classical, quantum, total)
in multipartite quantum states.

Two themes:

1. Separability/Entanglement
2. Classicality/Quantumness
 - ~ Non-disturbance/Disturbance
 - ~ Commutativity/Non-commutativity

Outline

1. States: Probabilities, Density Matrices
2. Separability/Entanglement
3. Classicality/Quantumness
4. Quantum Discord
5. Quantumness of Quantum Ensembles
6. No-cloning and No-broadcasting
7. Monogamy of Multipartite Correlations
8. Summary

1. States: Probabilities, Density Matrices

- A **classical state** is described by a probability distribution

$$\rho = (p_1, p_2, \dots, p_n),$$

which can also be represented by a **diagonal matrix**

$$\rho \sim \mathbf{p} = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{pmatrix}.$$

- Consider two classical states represented by diagonal matrices \mathbf{p} and \mathbf{q} . They always commute.
- Classicality \sim Commutativity

- Information content (Uncertainty) of a classical state $p = \{p_i\}$ is well quantified by the Shannon entropy (1948):

$$H(p) := - \sum_i p_i \log p_i.$$

- A **quantum state** is described by a density matrix (non-negative matrix with unit trace)

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}.$$

$$\text{tr}\rho = \sum_i \rho_{ii} = 1.$$

Here tr denotes the trace of a matrix.

Quantumness \sim Non-commutativity

- A single quantum state can always be regarded as classical in the sense that it can be diagonalized.
- But this is not the case for a set of several quantum states.

- Classical information content of a quantum state ρ is well quantified by the von Neumann entropy (1927):

$$S(\rho) := -\text{tr} \rho \log \rho.$$

- If $\rho = \sum_i p_i |i\rangle\langle i|$ is the spectral decomposition, then

$$S(\rho) = - \sum_i p_i \log p_i.$$

- Classical states can be naturally embedded in quantum states.
- Classical states are particular instances of quantum states.

Correlations are encoded in multipartite states

- **Classical case:** bivariate probability distribution p^{ab} with marginals p^a and p^b

The correlations in p^{ab} are usually quantified by the Shannon mutual information:

$$I(p^{ab}) = H(p^a) + H(p^b) - H(p^{ab})$$

- **Quantum case:** bipartite state ρ^{ab} , a density matrix in the tensor product Hilbert space $H^a \otimes H^b$, with marginals ρ^a and ρ^b

The (total) correlations in ρ^{ab} are usually quantified by the quantum mutual information:

$$I(\rho^{ab}) = S(\rho^a) + S(\rho^b) - S(\rho^{ab})$$

Fundamental difference between classical and quantum correlations

Perfect correlations

- Classical case: $p_{ij}^{ab} = p_i \delta_{ij}$

Shannon mutual information:

$$I(p^{ab}) = H(\{p_i\}).$$

- Quantum case: $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ with

$$|\Psi^{ab}\rangle = \sum_i \sqrt{p_i} |i\rangle \otimes |i\rangle$$

Quantum mutual information:

$$I(\rho^{ab}) = 2H(\{p_i\}).$$

Key issues: How to classify and quantify correlations in a bipartite state

- **Classification issue:** Different correlations, Separate total correlations into classical part and quantum part
- **Quantification issue:** Measures of various correlations

Two basic schemes

- Separability/Entanglement
- Classicality/Quantumness (of correlations)

2. Separability/Entanglement (Werner, 1989)

- A bipartite state ρ^{ab} shared by two parties a and b is **separable** if it can be represented as

$$\rho^{ab} = \sum_i p_i \rho_i^a \otimes \rho_i^b$$

with $\{p_i\}$ a probability distribution, $\{\rho_i^a\}$ and $\{\rho_i^b\}$ families of density matrices for parties a and b , respectively.

- Otherwise, ρ^{ab} is called **entangled**.

Detection and quantification of entanglement

- **Detection** (Hard problem): How to tell if a bipartite state is separable or entangled?

Various Bell inequalities

Peres' positive partial transposition

Methods based on uncertainty relations

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- **Quantification** (Complicated problem):
How to quantify the entanglement of a bipartite state?

Entanglement of formation

Entanglement cost

Relative entropy of entanglement

Squashed entanglement

Negativity

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A “paradox” for entanglement of formation

Werner state

$$w = \theta \frac{P_-}{d_-} + (1 - \theta) \frac{P_+}{d_+},$$

where P_- (P_+) is the projection from $C^d \otimes C^d$ to the anti-symmetric (symmetric) subspace of $C^d \otimes C^d$, and $d_{\pm} = \frac{d^2 \pm d}{2}$.

Entanglement of formation

$$E(w) = H\left(\frac{1}{2} - \sqrt{\theta(1-\theta)}\right).$$

Quantum mutual information

$$I(w) = \log \frac{2d^2}{(d^2 - d)^\theta (d^2 + d)^{1-\theta}} - H(\theta).$$

When d is sufficiently large,

$$E(w) > I(w).$$

quantum entanglement > total correlation?

Entanglement is not the only kind of quantum correlations

Certain quantum advantage is not based on quantum entanglement, but rather on separable states which still possess certain quantum correlations.

A. Datta, A. Shaji, C. M. Caves, Quantum discord and the power of one qubit, *Phys. Rev. Lett.* 100, 050502 (2008).

3. Classicality/Quantumness (of Correlations)

Piani et al., Luo, 2008

- A state ρ^{ab} shared by two parties a and b is called **classical** (w.r.t. correlations, or more precisely, classical-classical) if it is left undisturbed by certain local von Neumann measurement $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$, that is,

$$\rho^{ab} = \Pi(\rho^{ab}) := \sum_{ij} \Pi_i^a \otimes \Pi_j^b \rho^{ab} \Pi_i^a \otimes \Pi_j^b.$$

- Otherwise, it is called **quantum** (w.r.t. correlations).

- ρ^{ab} is called **classical-quantum**, if it is left undisturbed by a local von Neumann measurement $\Pi^a = \{\Pi_i^a\}$ for party a , that is,

$$\rho^{ab} = \Pi^a(\rho^{ab}) := \sum_i \Pi_i^a \otimes \mathbf{1}^b \rho^{ab} \Pi_i^a \otimes \mathbf{1}^b.$$

- ρ^{ab} is classical-classical iff

$$\rho^{ab} = \sum_{ij} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|.$$

Here $\{p_{ij}\}$ is a bivariate probability distribution, $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthogonal sets for parties a and b , respectively.

- ρ^{ab} is classical-quantum iff

$$\rho^{ab} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^b.$$

Separability vs. classicality

Li and Luo, Phys. Rev. A, 2008

- ρ^{ab} is separable iff it can be viewed as a local reduction of a classical state in a larger system:

$$\rho^{ab} = \text{tr}_{a'b'} \rho^{aa'bb'}.$$

Here $\rho^{aa'bb'}$ is a classical state (w.r.t. the cut $aa' : bb'$) on $(H^a \otimes H^{a'}) \otimes (H^b \otimes H^{b'})$.

4. Quantum Discord

- The quantum discord of ρ^{ab} is defined as

$$D(\rho^{ab}) := I(\rho^{ab}) - \sup_{\Pi^a} I(\Pi^a(\rho^{ab})).$$

Here $\Pi^a = \{\Pi_i^a\}$ is a von Neumann measurement for party a and

$$\Pi^a(\rho^{ab}) := \sum_i \Pi_i^a \otimes \mathbf{1}^b \rho^{ab} \Pi_i^a \otimes \mathbf{1}^b.$$

Vanishing of quantum discord

- The quantum discord of ρ^{ab} vanishes iff ρ^{ab} is a classical-quantum state.

- A. Shabani, D. A. Lidar, PRL, 2009

A bipartite state ρ^{ab} is classical-quantum iff for any unitary operator U^{ab} , the map

$$\text{tr}_a U^{ab} \rho^{ab} (U^{ab})^\dagger$$

is completely positive w.r.t. $\rho^a = \text{tr}_b \rho^{ab}$.

Luo, Quantum discord for two-qubit systems, Phys. Rev. A, 2008

- For two-qubit state

$$\rho^{ab} = \frac{1}{4}(\mathbf{1}^a \otimes \mathbf{1}^b + \sum_{j=1}^3 c_j \sigma_j^a \otimes \sigma_j^b),$$

we have

$$D(\rho^{ab}) = \frac{1}{4} \sum_{j=0}^3 \lambda_j \log \lambda_j - \frac{1+c}{2} \log(1+c) - \frac{1-c}{2} \log(1-c)$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$, and

$$\lambda_0 = 1 - c_1 - c_2 - c_3, \quad \lambda_1 = 1 - c_1 + c_2 + c_3$$

$$\lambda_2 = 1 + c_1 - c_2 + c_3, \quad \lambda_3 = 1 + c_1 + c_2 - c_3.$$

Sudden death of entanglement vs. robustness of quantum discord

- Yu and Eberly, **Sudden death of entanglement**, *Science*, 2009

During an evolution, entanglement may easily come to a death.

- **Robustness of quantum discord:**

Ferraro *et al.*, Phys. Rev. A, 2010

Almost all bipartite quantum states have a non-vanishing quantum discord.

For almost all initial states and almost all quantum evolution, the resulting final states have a non-vanishing quantum discord.

Werlang *et al.*, Phys. Rev. A, 2009

Quantum discord is more robust than entanglement.

Experimental investigation of classical and quantum correlations

J. S. Xu et al., Nature Communications **1**, 7 (2010).

An alternative quantum discord

- Geometric measure of quantum discord

$$D(\rho^{ab}) := \min_{\Pi^a} \|\rho^{ab} - \Pi^a(\rho^{ab})\|^2,$$

where the min is over von Neumann measurement $\Pi^a = \{\Pi_k^a\}$ on system H^a , and $\Pi^a(\rho^{ab}) := \sum_k (\Pi_k^a \otimes \mathbf{1}^b) \rho^{ab} (\Pi_k^a \otimes \mathbf{1}^b)$.

A formula

Let $\rho^{ab} = \sum_{ij} c_{ij} X_i \otimes Y_j$ be expressed in local orthonormal operator bases, then

$$D(\rho^{ab}) = \text{tr}(CC^t) - \max_A \text{tr}(ACC^tA^t),$$

where $C = (c_{ij})$, and the max is over $m \times m^2$ dimensional matrices $A = (a_{ki})$ such that $a_{ki} = \text{tr}|k\rangle\langle k|X_i$, and $\{|k\rangle\}$ is any orthonormal base for H^a .

In particular,

$$D(\rho^{ab}) \geq \text{tr}(CC^t) - \sum_{i=1}^m \lambda_i = \sum_{i=m+1}^{m^2} \lambda_i,$$

where λ_i are the eigenvalues of CC^t listed in decreasing order (counting multiplicity).

Observable correlations

- The observable correlations of ρ^{ab} is defined as

$$C(\rho^{ab}) := I(\rho^{ab}) - \sup_{\Pi} I(\Pi(\rho^{ab})).$$

Here $\Pi(\rho^{ab}) := \sum_{ij} \Pi_i^a \otimes \Pi_j^b \rho^{ab} \Pi_i^a \otimes \Pi_j^b$.

The quantity

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab})$$

may be interpreted as a measure of **quantum correlations** in ρ^{ab} .

Lindblad conjecture, 1991

- The Lindblad conjecture states that

$$C(\rho^{ab}) \geq Q(\rho^{ab}).$$

- Supporting evidence:

(1) It can happen $C(\rho^{ab}) > 0$ and $Q(\rho^{ab}) = 0$, but never $C(\rho^{ab}) = 0$ and $Q(\rho^{ab}) > 0$.

(2) For any classical state, we have $C(\rho^{ab}) = I(\rho^{ab})$ and $Q(\rho^{ab}) = 0$.

(3) For any pure state, we have $C(\rho^{ab}) = Q(\rho^{ab}) = \frac{1}{2}I(\rho^{ab})$.

Unfortunately, the Lindblad conjecture is false

Luo and Zhang, J. Stat. Phys. 2009

Counterexamples abounds!

5. Quantumness of Quantum Ensembles

- A quantum ensemble $\mathcal{E} = \{p_i, \rho_i\}$ consists of a family of quantum states ρ_i with corresponding probabilities p_i .
- How to distinguish two quantum ensembles?
- How to quantify the quantumness of a quantum ensemble?

- A measure of quantumness:

Let $\mathcal{E} = \{p_i, \rho_i\}$ be a quantum ensemble, then it can be canonically identified as a bipartite classical-quantum state

$$\rho_{\mathcal{E}} := \sum_i p_i |i\rangle\langle i| \otimes \rho_i.$$

Consequently, we may use the quantum correlations (e.g., quantum discord) to quantify the quantumness of the ensemble \mathcal{E} .

6. No-cloning and No-broadcasting

- Plausible observation:

Classical objects can be cloned and broadcast, while quantum objects cannot.

- Reasoning:

Otherwise, we could extract precise information from quantum system, and thus beat the Heisenberg uncertainty principle.

- Cloning and broadcasting demarcate the boundary between classical and quantum.

- A quantum state ρ in a Hilbert space H is clonable if there exists a quantum operation $\mathcal{E} : S(H) \rightarrow S(H) \otimes S(H)$ such that $\mathcal{E}(\rho) = \rho \otimes \rho$.
- A quantum state ρ in a Hilbert space H is broadcastable if there exists a quantum operation $\mathcal{E} : S(H) \rightarrow S(H) \otimes S(H)$ such that both the two marginal states of $\mathcal{E}(\rho)$ are ρ .

- Wootters and Zurek, A single quantum cannot be cloned, *Nature*, 1982
- Dieks, Communication by EPR device, *Phys. Lett. A*, 1982
- **Non-cloning Theorem:**
 1. An unknown quantum state cannot be cloned.
 2. A set of quantum states can be cloned iff they are classical.

No-broadcasting for non-commuting states, Barnum *et al.* Phys. Rev. Lett. 1996

A family of quantum states $\{\rho_i\}$ can be simultaneously broadcast iff the states are commutative.

Broadcasting for correlations

- The correlations in ρ^{ab} are locally broadcastable if there exist two operations $\mathcal{E}^a : \mathcal{S}(H^a) \rightarrow \mathcal{S}(H^{a_1} \otimes H^{a_2})$ and $\mathcal{E}^b : \mathcal{S}(H^b) \rightarrow \mathcal{S}(H^{b_1} \otimes H^{b_2})$ such that

$$I(\rho^{a_1 b_1}) = I(\rho^{a_2 b_2}) = I(\rho^{ab}).$$

Here $I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$ is the quantum mutual information,

$\rho^{a_1 a_2 b_1 b_2} := \mathcal{E}^a \otimes \mathcal{E}^b(\rho^{ab})$ and

$\rho^{a_1 b_1} := \text{tr}_{a_2 b_2} \rho^{a_1 a_2 b_1 b_2}$, $\rho^{a_2 b_2} := \text{tr}_{a_1 b_1} \rho^{a_1 a_2 b_1 b_2}$.

No-local-broadcasting for quantum correlations, Piani *et al.* Phys. Rev. Lett. 2008

The correlations in a bipartite state ρ^{ab} shared by two parties a and b can be locally broadcast iff the correlations are classical.

Unilocal broadcasting for correlations

- The correlations in ρ^{ab} is called locally broadcast by party a , if there exists an operation

$$\mathcal{E}^a : \mathcal{S}(H^a) \rightarrow \mathcal{S}(H^{a_1} \otimes H^{a_2})$$

such that $I(\rho^{a_1 b}) = I(\rho^{a_2 b}) = I(\rho^{ab})$. Here $\rho^{a_1 b} := \text{tr}_{a_2} \rho^{a_1 a_2 b}$, $\rho^{a_2 b} := \text{tr}_{a_1} \rho^{a_1 a_2 b}$, and $\rho^{a_1 a_2 b} := \mathcal{E}^a \otimes \mathcal{I}^b(\rho^{ab}) \in \mathcal{S}(H^{a_1} \otimes H^{a_2} \otimes H^b)$.

No-unilocal-broadcasting for quantum correlations, Luo, Lett. Math. Phys., 2010; Phys. Rev. A, 2010

The correlations in a bipartite state ρ^{ab} shared by two parties a and b can be locally broadcast by party a if and only if the correlations are classical-quantum (i.e., classical on party a).

Equivalence of the no-broadcasting theorems by Barnum *et al.* and by Piani *et al.*

7. Monogamy of Multipartite Correlations

Luo and Sun, Separability and entanglement in tripartite systems, Theor. Math. Phys. 2009

- Consider a tripartite state ρ^{abc} , if a and b are perfectly correlated, then a and c cannot be entangled.

8. Summary

- **The premise of correlations:**

A system in a pure state cannot have any correlation with other systems. In order for a system to establish correlations with other systems, it must be in a mixed state.

“水至清则无鱼，人至察则无徒。”

- **Distribution of correlations:**

Classical correlations can be arbitrarily distributed among systems.

Quantum correlations have monogamy: If system a has strong correlations with system b , then it cannot have strong correlations with any other system c . In particular, if system a is perfectly correlated with system b , then it cannot be entangled with any other system c .

- Species of correlations:

Scheme 1 Werner, 1989	Separable		Entanglement
Scheme 2 Piani et al. Luo, 2008	Classical	Quantum	
Scheme 3 Modi et al. 2010	Classical	Dissonance	Entanglement

- Quantification of correlations:

Various measures of correlations such as

Entanglement measures,

Quantum discord,

Observable correlations

Measurement-induced disturbance

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Thank you!