Nash-Kuiper Theorem on the density of isometric embeddings

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In this seminar I will review the proof of an improved version of the Nash-Kuiper Theorem given by S. Conti, C. De Lellis and L. Székelyhidi in 2009.

Given an n-dimensional Riemannian manifold (M, g), an isometric immersion is an injective map $u : (M, g) \to (\mathbb{R}^m, h)$ which preserves the metric, that is

$$u^\sharp h = g\,,$$

where h is the Euclidean metric on $\mathbb{R}^m$. We are interested in the case $m \le n+2$.

If $u \in \mathscr{C}^2$, some classical results in Differential Geometry force a "rigid" behaviour of the isometric embedding: the Gaussian curvature is preserved and u is necessarily a roto-translation.

On the contrary, if we are allowed to consider less regular maps, then the following theorem holds:

Theorem 1 (Conti – De Lellis – Székelyhidi)**.** *Let $u_0 : M \to \mathbb{R}^{n+1}$ be a $\mathscr{C}^{1,\alpha}$-embedding with the following property*

$$g - u_0^\sharp h \ge 0\,.$$

Then, for every $\varepsilon > 0$, there exists an isometric embedding $u \in \mathscr{C}^{1,\alpha}$ such that

$$\|u - u_0\|_{\mathscr{C}^0} < \varepsilon\,.$$

This theorem improves the celebrated result by J. Nash in 1954 (with $m = n+2$ and $u \in \mathscr{C}^1$).

Nash's proof has become even more interesting for its connection with Gromov's h-principle: indeed, Nash-Kuiper Theorem is one of the main examples of convex integration techniques applied to a density problem for partial differential relations.