On the condition number of the polynomial eigenvalue problem with random inputs

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A (real) *polynomial eigenvalue* of a $(d+1)$-uple of $n \times n$ (real) matrices $A = (A_0, \ldots, A_d)$ is a projective root $x = [\alpha : \beta] \in \mathbb{R}\mathrm{P}^1$ of $\det(\beta^d A_0 + \alpha\beta^{d-1} A_1 + \cdots + \alpha^d A_d) = 0$. The "worst" infinitesimal change $\mu(A, x)$ in $x \in \mathbb{R}\mathrm{P}^1$ under an infinitesimal perturbation of A is called *the local condition number of A at x*. Then *the condition number* $\mu(A)$ of A is the sum $\mu(A) = \sum \mu(A, x)$ over all polynomial eigenvalues of A. We provide explicit formulas for the expected condition number $\mathbb{E}\,\mu(A)$ when the matrices $A_0, A_1, \ldots, A_d$ are drawn from various random matrix ensembles. For example, if $A_0, A_1, \ldots, A_d$ are independent matrices whose entries are independent standard Gaussian variables we have

$$\mathbb{E}\,\mu(A) = \pi \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{(d+1)n^2}{2}\right)}{\Gamma\left(\frac{(d+1)n^2-1}{2}\right)} = \frac{\pi}{2}\sqrt{d+1}\, n^{3/2} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \; n \to \infty.$$

This formula establishes the *asymptotic square root law* for the polynomial eigenvalue problem, i.e., the answer (asymptotically, when $n \to \infty$) is the square root of the answer to the analogous problem over the complex numbers investigated in a recent paper by D. Armentano and C. Beltran.