On the geometry of the set of symmetric matrices with repeated eigenvalues

Let $\Delta_n \subset \mathrm{P}(\mathrm{Sym}(n, \mathbb{R}))$ denote the projective variety of real symmetric matrices of size $n \times n$ having repeated eigenvalues. We prove that the volume of Δ_n satisfies:

$$\frac{|\Delta_n|}{|\mathbb{R}\mathrm{P}^{N-3}|} = \binom{n}{2},$$

where $N = \binom{n+1}{2}$ is the dimension of the space of real symmetric matrices of size $n \times n$. Equivalently, the average number of symmetric matrices having repeated eigenvalues in a uniformly distributed projective 2-plane $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathrm{P}(\mathrm{Sym}(n, \mathbb{R}))$ equals $\binom{n}{2}$. A sharp deterministic upper bound on the number of symmetric matrices with repeated eigenvalues in a generic L is $\binom{n+1}{3}$, the degree of the variety Δ_n.

(Joint work with P. Breiding and A. Lerario)