On necessary and sufficient conditions for entire functions to belong to the Laguerre-Pólya class in terms of their Taylor coefficients

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We discuss new conditions for the entire functions with positive Taylor coefficients to belong to the Laguerre–Pólya class. For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ let us define the second quotients of Taylor coefficients as $q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, n \geq 2$.

The *partial theta-function*, $g_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}$, $a > 1$, was studied by many authors. In [1] it is proved that there exists a constant $q_\infty, q_\infty \approx 3.23363666$, such that the partial theta-function (and all its odd Taylor sections) belongs to the Laguerre-Pólya class if and only if $a^2 \geq q_\infty$.

We obtained new necessary and sufficient conditions on the Taylor coefficients of entire functions to belong to the Laguerre-Pólya class. The following theorem illustrates a sufficient condition for the case when $q_n(f)$ are decreasing in n.

Theorem (T. H. Ng, Anna Vishnyakova, [2]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k, be an entire function. Suppose that $q_n(f)$ are decreasing in n, i.e. $q_2(f) \geq q_3(f) \geq q_4(f) \geq \ldots$, and $\lim\limits_{n\to\infty} q_n(f) = b \geq q_\infty$. Then all the zeros of f are real and negative, in other words $f \in \mathcal{L} - \mathcal{P}$.*

It is easy to see that, if only the estimation of $q_n(f)$ from below is given and the assumption of monotonicity is omitted, then the Hutchinson's constant 4 in $q_n(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$.

The following result provides a necessary condition for an entire function with positive coefficients and with the increasing second quotients of Taylor coefficients to belong to the Laguerre-Pólya class.

Theorem (T. H. Ng, Anna Vishnyakova, [3]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all* k, *be an entire function such that the quotients* $q_n(f)$ *are increasing in* n. *If* f *belongs to the Laguerre-Pólya class, then* $\lim\limits_{n\to\infty} q_n(f) \geq q_\infty$.

We also present other new related necessary and sufficient conditions.

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