Title "An optimal variance estimate in stochastic homogenization"

Felix Otto, Institut für Angewandte Mathematik, Universität Bonn

We consider a discrete elliptic equation with random coefficients A, which (to fix ideas) are identically distributed and independent from grid point to grid point $x \in \mathbb{Z}^d$. On scales large w. r. t. the grid size (i. e. unity), the solution operator is known to behave like the solution operator of a (continuous) elliptic equation with constant deterministic coefficients. These symmetric "homogenized" coefficients A_{hom} are characterized by

$$\xi \cdot A_{hom}\xi \;=\; \langle ((\xi + \nabla\phi) \cdot A(\xi + \nabla\phi))\,(0)\rangle, \quad \xi \in \mathbb{R}^d,$$

where the random field ϕ is the unique stationary solution of the "corrector problem"

$$-\nabla \cdot A(\xi + \nabla\phi) \;=\; 0$$

and $\langle\cdot\rangle$ denotes the ensemble average.

It is known ("by ergodicity") that the above ensemble average of the energy density $e = (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)$, which is a stationary random field, can be recovered by a system average. We quantify this by proving that the variance of a spatial average of e on length scales L is estimated as follows:

$$\mathrm{var}\left[\sum_{x \in \mathbb{Z}^d} \eta_L(x)\, e(x)\right] \;\lesssim\; L^{-d},$$

where the averaging function (i. e. $\sum_{x\in\mathbb{Z}^d} \eta_L(x) = 1$, $\mathrm{supp}\eta_L \subset [-L, L]^d$) has to be smooth in the sense that $|\nabla\eta_L| \lesssim L^{-1}$. In two space dimensions (i. e. $d = 2$), there is a logarithmic correction.

In other words, smooth averages of the energy density e behave like as if e would be independent from grid point to grid point (which it is not for $d > 1$). This result is of practical significance, since it allows to estimate the error when numerically computing A_{hom}.

This is joint work with Antoine Gloria, INRIA Lille.