When are multidegrees positive?

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Degree

Let $X \subset \mathbb{P}_k^m$ be a projective variety over a field k. There are two important integers that provide information about X: **dimension** d and **degree** e.

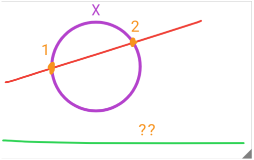
Geometrically, the degree is equal to the number of points in the intersection of X with a *generic* linear space L codimesion d.

Algebraically, the degree is $d!$ times the leading coefficient of the Hilbert polynomial of X.

The geometric definition need algebraically closed, as the next example will show.

Example

Let $X \subset \mathbb{P}^2_k$ be the zeros of the polynomial $x^2 + y^2 = z^2$, i.e., the circle. If $k = \mathbb{R}$ then there are generic lines with zero points in the intersection.



Algebraically, we have need to compute the Hilbert polnomial of the standard graded ring $R = k[x, y, z]/\langle x^2 + y^2 - z^2 \rangle$. In this example we have $H(R, 0) = 1, H(R, 1) = 3, H(R, 2) = 5$ and in general

$$P_R(t) = \binom{t+2}{2} - \binom{t}{2} = 2t + 1 = \frac{\mathbf{2}}{1!} t^1 + 1$$

Extra assumptions

Degrees only depend on the **reduced** structure. Furthermore, from the geometric intuition is clear that degree is **additive** over the irreducible components of the variety. For these reasons we assume X is **irreducible** and on the algebra side we work with a standard graded **domain** R which is a homogeneous quotient of $k[x_1, \cdots, x_m]$.

Question

For any field k and any m does there exists $X \subset \mathbb{P}_k^m$ irreducible with any given degree and dimension (less than m)?

Multidegrees

Let $X \subset \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^m$ be an irreducible variety over a field k. As a variety, X has some **dimension** d but now there is no single degree. Instead, we intersect with products of linear spaces whose codimensions add d.

Multidegrees

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Geometrically, for any $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$ with $n_1 + n_2 + n_3 = d$ we define $\deg^{\mathbf{n}}(X)$ as the number of points in the intersection of X with a product $L_1 \times L_2 \times L_3$, where $L_i \subseteq \mathbb{P}^m$ is a general linear subspace of codimension n_i for each $1 \leq i \leq 3$.

Algebraically, we have a ring R that is a multihomogeneous quotient of $k[x_{ij} : 0 \leq i \leq m, 1 \leq j \leq 3]$ with standard grading $\deg(x_{ij}) = \mathbf{e}_j \in \mathbb{N}^3$. Then multidegrees appear in the multivariate Hilber polynomial when expanded in certain basis:

$$P_R(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{N}^3} \deg^{\mathbf{n}} \binom{t_1 + n_1}{n_1} \binom{t_2 + n_2}{n_2} \binom{t_3 + n_3}{n_3}$$

Chow rings

The Chow ring of $\mathbb{P} = \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_p}$ is given by

$$A^*(\mathbb{P}) = \frac{\mathbb{Z}[H_1, \ldots, H_p]}{\left(H_1^{m_1+1}, \ldots, H_p^{m_p+1}\right)}$$

where H_i represents the class of the inverse image of a hyperplane of $\mathbb{P}_k^{m_i}$ under the natural projection $\Pi_i : \mathbb{P} \to \mathbb{P}_k^{m_i}$.
The class of the cycle associated to X coincides with

$$[X] \;=\; \sum_{\substack{0 \leq n_i \leq m_i \\ |\mathbf{n}| = r}} \deg_{\mathbb{P}}^{\mathbf{n}}(X)\, H_1^{m_1 - n_1} \cdots H_p^{m_p - n_p} \;\in A^*(\mathbb{P}).$$

Question

Which classes are representable by irreducible varieties?

Example

Consider the affine toric variety $Z_A \subset \mathbb{A}^6$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 1 & 1 & 5 \end{bmatrix}.$$

By definition Z_A is the closure of the image of the monomial map given by

$$(\mathbb{C}^*)^2 \longrightarrow \mathbb{A}^6$$
$$(t_1, t_2) \mapsto (t_1^1 t_2^2, t_1^2 t_2^4, t_1 t_2^2, t_1^2 t_2, t_1 t_2, t_2^5)$$

Using M2 we can compute its toric ideal. This ideal is a binomial prime ideal so that Z_A is irreducible. We can now compute its closure Y_A on $(\mathbb{P}^2)^3$ by homogenizing. Then $Y_A \subset (\mathbb{P}^2)^3$ is an irreducible subvariety. Using M2 to compute its multidegree we obtain:

$$5H_3^2 + 10H_2H_3 + 3H_2^2 + 12H_1H_3 + 6H_1H_2.$$

When are multidegrees positive?

Let k be an arbitrary field, $\mathbb{P} = \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_p}$ be a multiprojective space over k, and $X \subseteq \mathbb{P}$ be a closed irreducible subscheme of $\mathbb{P}$.

Theorem

Let $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$ be such that $n_1 + \cdots + n_p = \dim(X)$. Then, $\deg_{\mathbb{P}}^{\mathbf{n}}(X) > 0$ if and only if for each $\mathfrak{J} = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, p\}$ the following inequality holds

$$n_{j_1} + \cdots + n_{j_k} \leq \dim\left(\Pi_{\mathfrak{J}}(X)\right)$$

Furthermore, the function $r_X : 2^{[p]} \to \mathbb{Z}$ defined by $r_X(\mathfrak{J}) = dim\big(\Pi_{\mathfrak{J}}(X)\big)$ is submodular.

Theorem

When X is irreducible, the set

$$\mathrm{MSupp}_{\mathbb{P}}(X) = \left\{\mathbf{n} \in \mathbb{N}^p \mid \deg_{\mathbb{P}}^{\mathbf{n}}(X) > 0\right\},$$

is a discrete polymatroid.

Submodular functions

A function $f : 2^{[p]} \to \mathbb{R}$ is **submodular** if
$f(\mathfrak{I}) + f(\mathfrak{J}) \geq f(\mathfrak{I} \cap \mathfrak{J}) + f(\mathfrak{I} \cup \mathfrak{J})$, for any $\mathfrak{I}, \mathfrak{J} \subset [p]$.

Submodular functions abound.

Linear algebra

Let $\{v_1, \cdots, v_p\}$ be a set of p vectors. The function $f(\mathfrak{I}) := \mathrm{rank}\{v_i : i \in \mathfrak{I}\}$ is submodular.

Graph cuts

Let G be a graph on the ground set $[p]$. For any subset $\mathfrak{I}$ of vertices, we can define $f(\mathfrak{I})$ as the size of the cut induced by $\mathfrak{I}$. This is the amount of edges with one vertex in $\mathfrak{I}$ and the other in $\overline{\mathfrak{I}}$ (the complement). This cut size function is submodular.

Probability

Let $(A_1, \cdots, A_p)$ a set of random events (i.e. $0-1$ random variables). The function $f(\mathfrak{I}) := \text{Prob}(A_i = 1 : \forall i \in \mathfrak{I})$ is submodular.

Chow polymatroids

We call a submodular function f **Chow** if there exists an embedded multiprojective variety X such that $f(\mathfrak{I}) = \dim(\Pi_{\mathfrak{I}}(X))$

Question

Can we classify Chow functions?

Chow functions are algebraic, which means that some functions are not Chow (e.g. Vamos Matroid). On the other hand, in characteristic zero algebraic and linear coincide.

Theorem

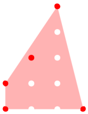
In characteristic zero all Chow functions are linear. We have a explicit construction for each possible Chow function.

Newton polytopes

Let $p = \sum c_{\mathbf{a}} x^{\mathbf{a}} \in \mathbb{Z}[x_1, \cdots, x_p]$ be a polynomial. Then the **Newton polytope** $\mathrm{Newton}(p)$ is $\mathrm{conv}(\mathbf{a} \in \mathbb{Z}^p : c_{\mathbf{a}} \neq 0)$. The polynomial is said to have Saturated Newton Polytope (SNP) property if $\mathbf{a} \in \mathrm{Newton}(p) \iff c_{\mathbf{a}} \neq 0$ for all $\mathbf{a} \in \mathbb{Z}^p$.

Example

Let $p = 2x^3 - x^2y^4 + 7xy^2 + 6y + 3 \in k[x, y]$. Then p does not posses SNP.

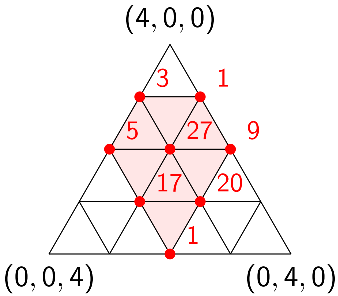


Generalized permutohedra

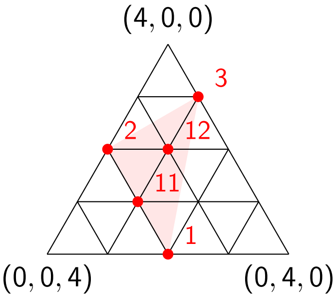
The Newton polytope of multidegrees are base polymatroid polytopes, which also appear in the literature as **generalized permutohedra**. This family is charaterized by the property that all edges are parallel to edges of the standard simplex $\Delta_n := \mathrm{conv}(\mathbf{e}_i : i \in [n])$.

Furthermore, multidegrees have the SNP property.

Representable classes



Representable



Not Representable

Applications

We will see next two applications: Schubert polynomials and Mixed volumes.

Schubert polynomials

The **Schubert polynomial** $\mathfrak{S}_\pi$ is the class of the Schubert variety $[Y_\pi]$ in the cohomology ring of the flag variety $\mathrm{Fl}(n)$ with the following presentation

$$R_n = k[x_1, \cdots, x_n]/\langle \text{symmetric polynomials w/o constant term.} \rangle$$

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Combinatorially they can be defined recursively as

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$
$$\mathfrak{S}_{ws_i} = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}}, \quad \ell(ws_i) < \ell(w),$$

where $w_0 = n(n-1)\cdots 321$ is the *longest* element and $s_i := (i, i+1)$.

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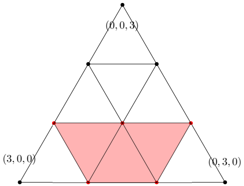
Example

$$\mathfrak{S}_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3.$$

Schubert polynomials

For the example before,
$\mathfrak{S}_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$, we have



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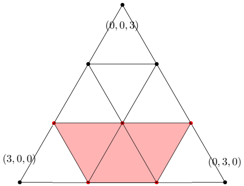

Theorem (Fink-Mezaros-St.Dizier '18)

The Newton polytope of a Schubert polynomial is a generalized permutohedron and it is **saturated**.

We provide an alternative proof.

Mixed volumes

Let $P_1, \cdots, P_n$ be polytopes in $\mathbb{R}^m$ with $n \leq m$. Consider

$$\mathrm{Vol}(x_1 P_1 + x_2 P_2 + \cdots + x_n P_n),$$

this is an **homogeneous** polynomial on x_i of degree m and the coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ is denoted $\mathrm{MV}(P_1^{a_1}, \cdots, P_n^{a_n})$ are called the **mixed volumes**.

When are mixed volumes positive?

Theorem(Minkowski)

The following are equivalent:

- $\mathrm{MV}(P_1^{a_1}, \cdots, P_n^{a_n}) > 0$.
- There exist line segments $S_{i,1}, S_{i,2}, \ldots, S_{i,a_i} \subseteq P_i$, for every i, such that $\{S_{i,j}\}_{1 \leq i \leq p, 1 \leq j \leq a_i}$ has segments in d linearly independent directions.
- $\sum_{i=1}^{n} a_i = d$ and $\sum_{i \in \mathfrak{J}} a_i \leq \dim \left(\sum_{i \in \mathfrak{J}} P_i \right)$ for every subset $\mathfrak{J} \subseteq [n]$.

We provide an alternative proof.

Speculative remarks

- Cohen-Macaulay case.
- Non standard case.
- Other Chow rings? Cox ring of toric varieties?.

The big question is still which classes are representable. (related to log-concavity, Lorentzian polynomials, etc.)

The End