Non-integer multigraded algebra

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Nonlinear Algebra Meeting Online

Max Planck Institute für Mathematik in den Naturwissenschaften

17 November 2020

Outline

1. Polynomials with real exponents
2. Poset modules
3. Partially ordered groups
4. Tameness
5. Primary decomposition
6. Coprimary modules
7. Future directions

Polynomials with real exponents

Def. The ring of real-exponent polynomials in n variables over field $\Bbbk$ is

$$\Bbbk[\mathbb{R}_+^n] = \bigoplus_{\mathbf{a}\in\mathbb{R}_+^n} \Bbbk \cdot \{\mathbf{x}^{\mathbf{a}}\} \qquad \text{with} \qquad \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}},$$

a monoid algebra.

Example. $(x^{\sqrt{2}} + y^{\pi})(xy^2 - z) = x^{1+\sqrt{2}}y^2 + xy^{2+\pi} - x^{\sqrt{2}}z - y^{\pi}z$

An $\mathbb{R}^n$-graded module over $\Bbbk[\mathbb{R}_+^n]$ is

$$M = \bigoplus_{\mathbf{a}\in\mathbb{R}^n} M_{\mathbf{a}} \qquad \text{with action} \qquad \mathbf{x}^{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}.$$

Examples. monomial ideal $I = \langle \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in A\rangle$ for some $A \subseteq \mathbb{R}_+^n$

1. $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \mid a_i > 0 \;\forall\, i\,\rangle = \mathfrak{m} =$ maximal monomial ideal
   - countably generated
   - no minimal generating set
2. $I = \langle \mathbf{x}^{\mathbf{a}} \mid a_1 + \cdots + a_n = 1 \text{ and } a_i \geq 0 \;\forall\, i\,\rangle$
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Polynomials with real exponents

Def. The ring of real-exponent polynomials in n variables over field $\Bbbk$ is

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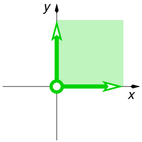

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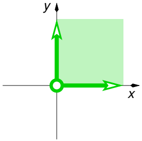

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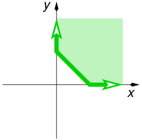

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- discretely generated ideals [Ingebretson, Sather-Wagstaff, Andersen 2013–2015]
- Descartes' rule of signs: some multivariate cases [Bihan, Rojas, Sottile 2008]
- almost no commutative or homological algebra in general

Good news: analogues exist for

- presentation
- syzygy theorem
- primary decomposition

Bad news

- finite vs. free: these are incompatible
- minimality: real exponents ⇒ subtle

Developments: commutative algebra on posets

1. theory of tame modules to replace "noetherian" [arXiv:2008.00063]
2. primary decomposition over partially ordered abelian group [arXiv:2008.00093]
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Def. *Q-module* over the poset Q:

- Q-graded vector space $M = \bigoplus_{q \in Q} M_q$ over the field $\Bbbk$ with
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Examples

- $Q = \mathbb{Z}^n \Leftrightarrow M = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
- $Q = \mathbb{R}^n \Leftrightarrow M = \mathbb{R}^n$-graded $\Bbbk[\mathbb{R}^n_+]$-module
- wing veins: $Q = \mathbb{Z}^2$ (two discrete parameters)
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Motivation. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q \Rightarrow$ partial ordering on Q via $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has *persistent homology* the Q-module H with $H_q = H(X_q; \Bbbk)$

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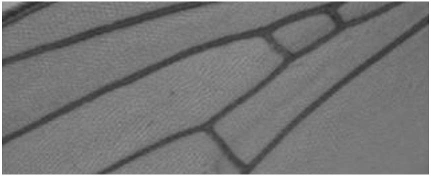

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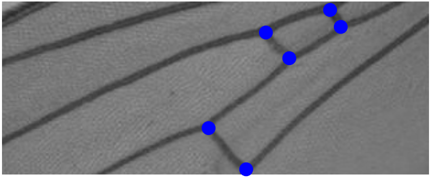

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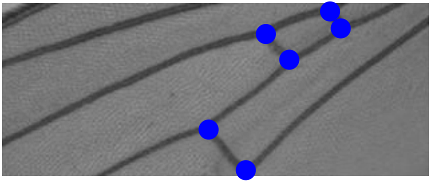

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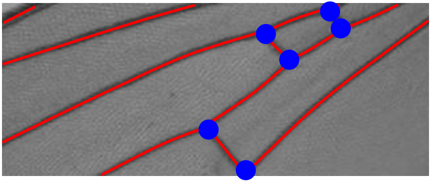

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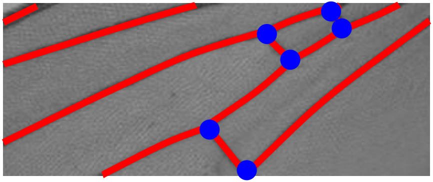

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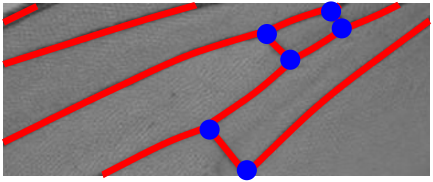

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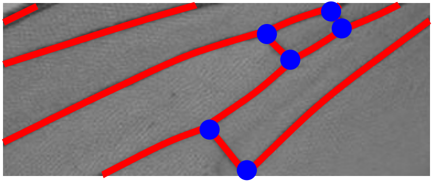

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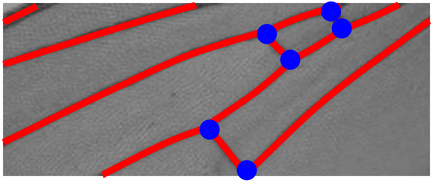

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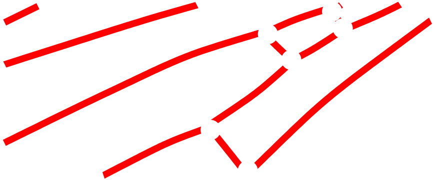

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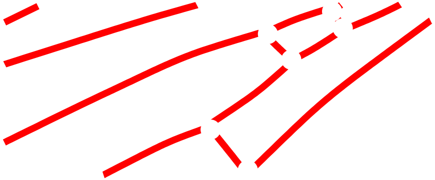

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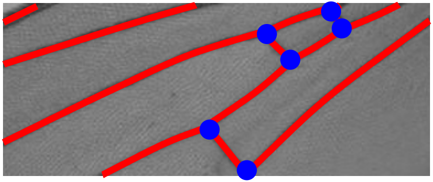

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Monomial ideals: combinatorial primary decomposition

- teases apart groups of monomials parallel to the coordinate planes
- or parallel to faces in affine semigroup rings

Binomial ideals [Eisenbud–Sturmfels 1996, Kahle–M 2014, Kahle–M–O'Neill 2016]
Combinatorial primary decomposition requires positive multigrading
$\leftrightarrow$ cone of positive elements greater than 0.

Question. Over which posets does monomial primary decomposition work?

Remark. Homological algebra generalizes to modules over any poset
[syzygy theorem, arXiv:2008.00063]

Answer 1 [–]. Need at least

- faces ($\leftrightarrow$ prime ideals)
- localization (without altering ambient poset)

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- localization (without altering ambient poset) $\Rightarrow$ poset $\subseteq$ group

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Lemma. Q-module $\leftrightarrow$ Q-graded module over $\Bbbk[Q_+]$

Partially ordered groups

Monomial ideals: combinatorial primary decomposition

- teases apart groups of monomials parallel to the coordinate planes
- or parallel to faces in affine semigroup rings

Binomial ideals [Eisenbud–Sturmfels 1996, Kahle–M 2014, Kahle–M–O'Neill 2016]
Combinatorial primary decomposition requires positive multigrading
$\leftrightarrow$ cone of positive elements greater than 0.

Question. Over which posets does monomial primary decomposition work?

Remark. Homological algebra generalizes to modules over any poset
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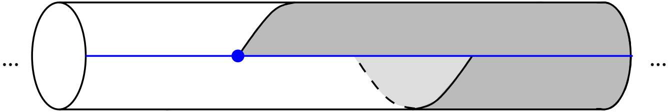

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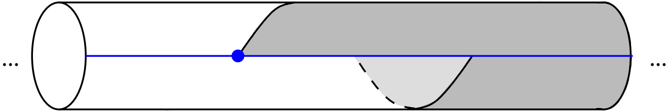

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Finiteness conditions: • $\mathbb{Z}^n$-modules: finitely generated $\Leftrightarrow$ noetherian
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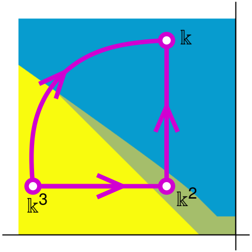

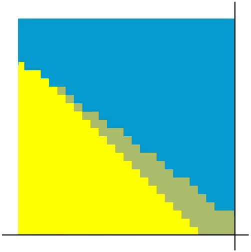

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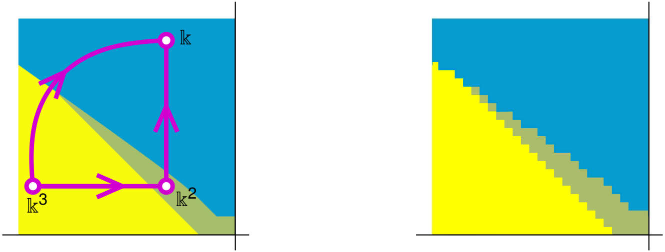

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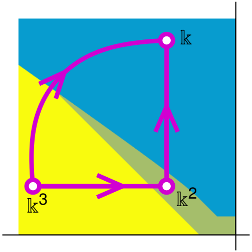

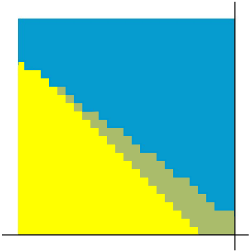

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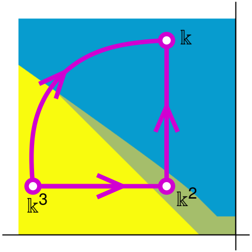

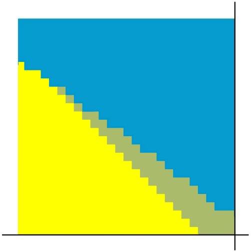

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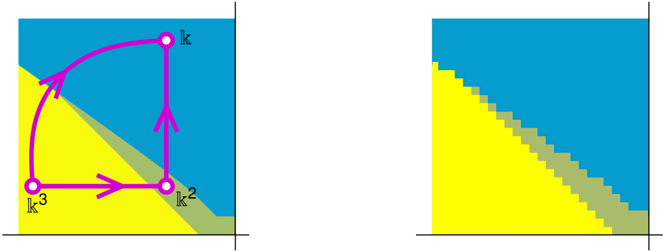

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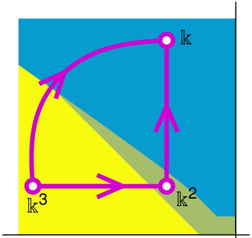

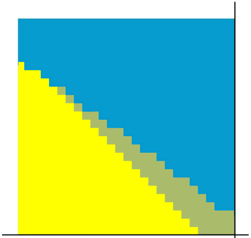

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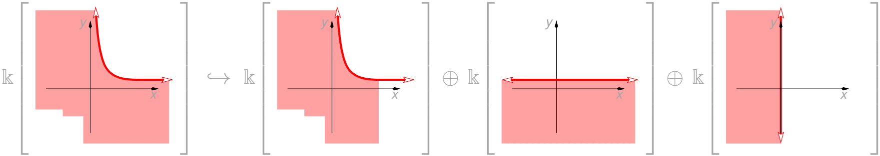

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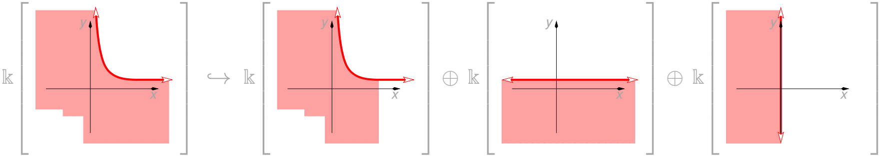

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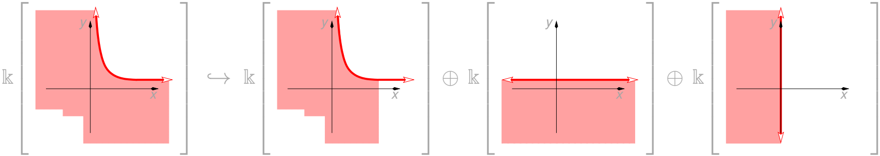

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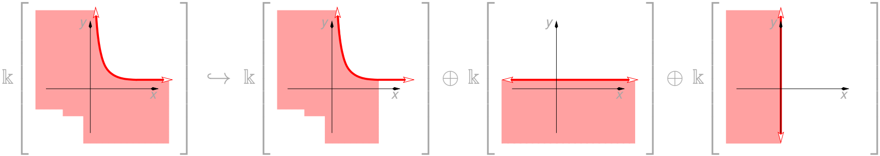

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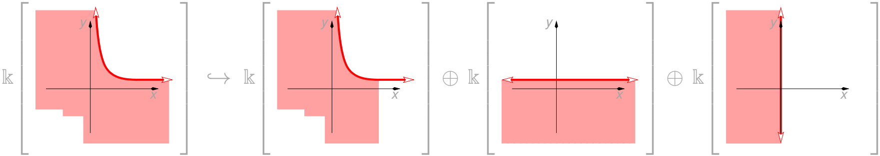

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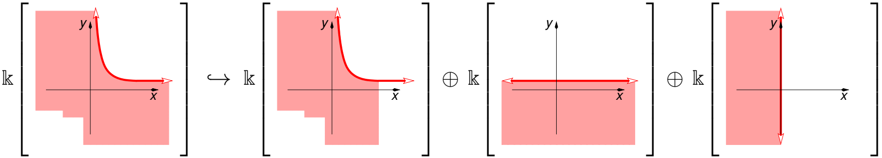

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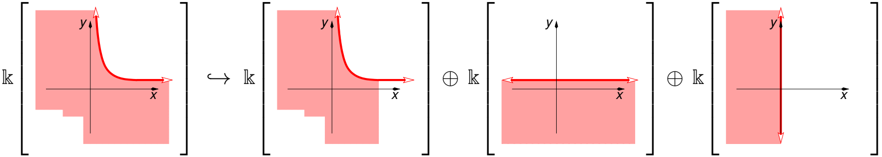

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Remark. $\Leftrightarrow 0 = \bigcap_\tau \ker(M \to M_\tau)$ with $\ker(M \to M_\tau)$ primary in M

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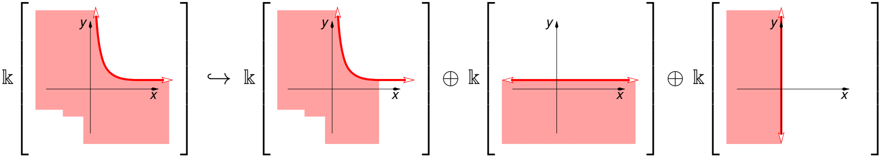

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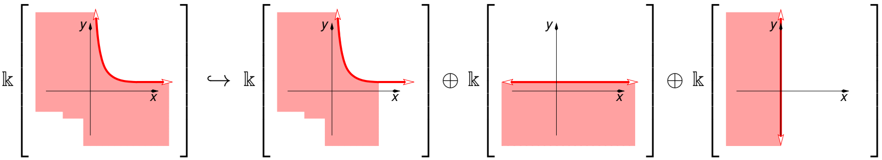

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Remark. Minimality needs $\mathbb{Z}^n$ or $\mathbb{R}^n$ [arXiv:2008.03819] or other geometric control

Coprimary modules

Def [–]. In a partially ordered group Q, a ray of Q_+ (or of Q itself) is a face that is totally ordered as a partially ordered submonoid of Q.

Answer 3 [–]. Q is closed if the complement $Q_+ \smallsetminus \tau$ of each face τ is generated as an ideal of Q_+ by $\rho \smallsetminus \{\mathbf{0}\}$ for the rays $\rho \not\subseteq \tau$.

Def [–]. If Q is closed and τ is a face, then a Q-module element is

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A Q-module is τ-coprimary if every element divides a coprimary element.

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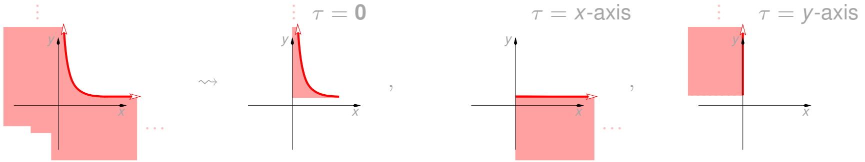

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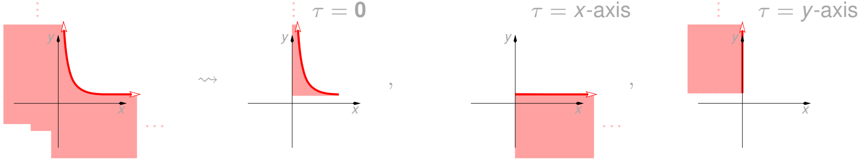

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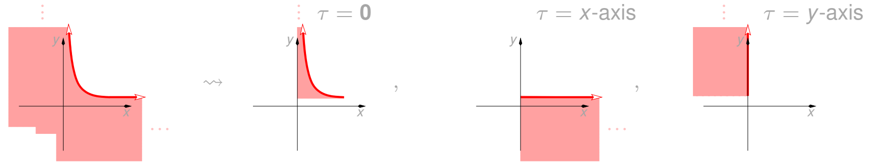

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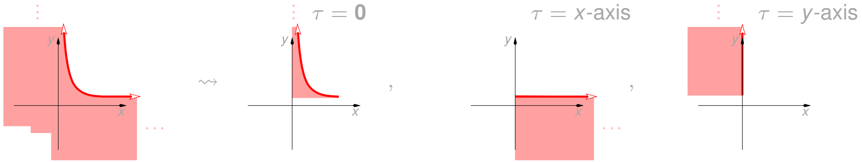

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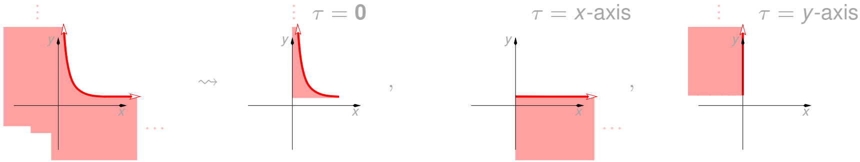

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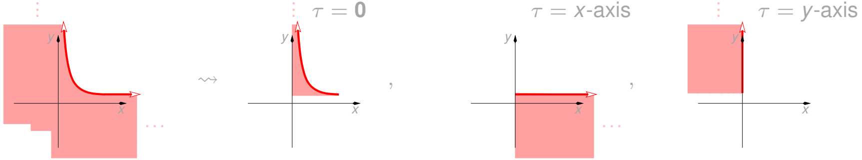

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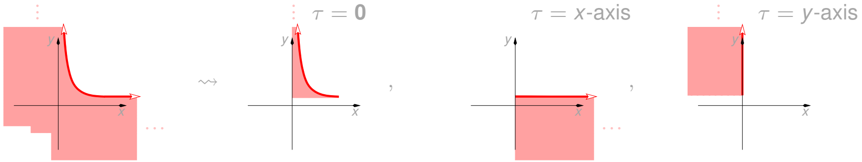

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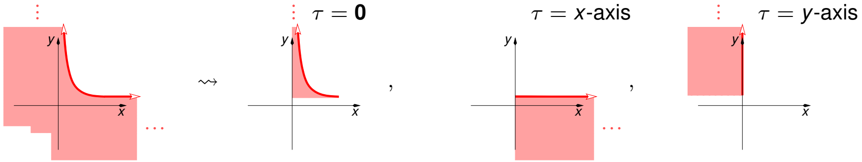

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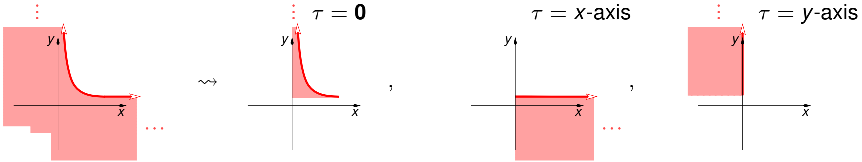

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- indecomposable decomposition $\leftarrow$ in progress with undergrad Joey Li

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- analogue of Hochster's formula for real-exponent monomial ideals
- Does $\mathbb{k}[\mathbb{R}^n_+]$ have finite global dimension in the usual sense?
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