Non-integer multigraded algebra

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Nonlinear Algebra Meeting Online
Max Planck Institute für Mathematik in den Naturwissenschaften

17 November 2020
Outline

1. Polynomials with real exponents
2. Poset modules
3. Partially ordered groups
4. Tameness
5. Primary decomposition
6. Coprimary modules
7. Future directions
Polynomials with real exponents

**Def.** The ring of real-exponent polynomials in $n$ variables over field $k$ is

$$k[\mathbb{R}^n_+] = \bigoplus_{a \in \mathbb{R}^n_+} k \cdot \{x^a\} \quad \text{with} \quad x^a x^b = x^{a+b},$$

a monoid algebra.

Example. $(x^{\sqrt{2}} + y^\pi)(xy^2 - z) = x^{1+\sqrt{2}} y^2 + x y^{2+\pi} - x^{\sqrt{2}} z - y^\pi z$

An $\mathbb{R}^n$-graded module over $k[\mathbb{R}^n_+]$ is

$$M = \bigoplus_{a \in \mathbb{R}^n} M_a \quad \text{with action} \quad x^a M_b \subseteq M_{a+b}.$$

Examples. monomial ideal $I = \langle x^a \mid a \in A \rangle$ for some $A \subseteq \mathbb{R}^n_+$

1. $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \mid a_i > 0 \ \forall \ i \rangle = \mathfrak{m} = \text{maximal monomial ideal}$
   - countably generated
   - no minimal generating set

2. $I = \langle x^a \mid a_1 + \cdots + a_n = 1 \text{ and } a_i \geq 0 \ \forall \ i \rangle$
   - uncountably generated
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**Def.** The ring of real-exponent polynomials in $n$ variables over field $\mathbf{k}$ is

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   - countably generated
   - no minimal generating set

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Context

Prior research

- finitely presented \(\Rightarrow\) Krull–Schmidt: \(\bigoplus\) indecomposables [Lesnick 2011]
- discretely generated ideals [Ingebretson, Sather-Wagstaff, Andersen 2013–2015]
- Descartes’ rule of signs: some multivariate cases [Bihan, Rojas, Sottile 2008]
- almost no commutative or homological algebra in general

Good news: analogues exist for

- presentation
- syzygy theorem
- primary decomposition

Bad news

- finite vs. free: these are incompatible
- minimality: real exponents \(\Rightarrow\) subtle

Developments: commutative algebra on posets

2. primary decomposition over partially ordered abelian group [arXiv:2008.00093]
5. interpretation in terms of constructible sheaves via [Kashiwara–Schapira 2018]
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Developments: commutative algebra on posets

1. theory of tame modules to replace “noetherian” $\leftarrow$ a little today!
2. primary decomposition over partially ordered abelian group $\leftarrow$ a lot today!
5. interpretation in terms of constructible sheaves via [Kashiwara–Schapira 2018]
Modules over posets

Def. \textit{Q-module} over the poset \( Q \):

- \( Q \)-graded vector space \( M = \bigoplus_{q \in Q} M_q \) over the field \( \mathbf{k} \) with

- homomorphism \( M_q \rightarrow M_{q'} \) whenever \( q \preceq q' \) in \( Q \) such that

- \( M_q \rightarrow M_{q''} \) equals the composite \( M_q \rightarrow M_{q'} \rightarrow M_{q''} \) whenever \( q \preceq q' \preceq q'' \)

Examples

- \( Q = \mathbb{Z}^n \leftrightarrow M = \mathbb{Z}^n \)-graded \( \mathbf{k}[x_1, \ldots, x_n] \)-module
- \( Q = \mathbb{R}^n \leftrightarrow M = \mathbb{R}^n \)-graded \( \mathbf{k}[\mathbb{R}_+^n] \)-module
- wing veins: \( Q = \mathbb{Z}^2 \) (two discrete parameters)
- wing veins: \( Q = \mathbb{R}^2 \) (two continuous parameters)

Motivation. Topological space \( X \) filtered by set \( Q \) of subspaces: \( X_q \subseteq X \) for \( q \in Q \Rightarrow \) partial ordering on \( Q \) via \( X_q \subseteq X_{q'} \Leftrightarrow q \preceq q' \)

Def. \( \{X_q\}_{q \in Q} \) has \textbf{persistent homology} the \( Q \)-module \( H \) with \( H_q = H(X_q; \mathbf{k}) \)

Real exponents | Poset modules | Partially ordered groups | Tameness | Primary decomposition | Coprimary modules | Future
**Modules over posets**

**Def.** *Q*-module over the poset *Q*: (think *Q* = ℤ<sup>*</sup>n or ℜ<sup>n</sup>)

- *Q*-graded vector space *M* = ⊕<sub>*q*∈*Q* *M*<sub>*q*> over the field *k* with
- homomorphism *M*<sub>*q*> → *M*<sub>*q′*> whenever *q* ≤ *q′* in *Q* such that
- *M*<sub>*q*> → *M*<sub>*q″*> equals the composite *M*<sub>*q*> → *M*<sub>*q′*> → *M*<sub>*q″*> whenever *q* ≤ *q′* ≤ *q″*

**Examples**

- *Q* = ℤ<sup>n</sup> ⇔ *M* = ℤ<sup>n</sup>-graded *k*[*x*<sub>1</sub>, . . . , *x*<sub>*n*<sub>]-module
- *Q* = ℜ<sup>n</sup> ⇔ *M* = ℜ<sup>n</sup>-graded *k*[ℜ<sup>n</sup>]<sub>+</sub>-module
- wing veins: *Q* = ℤ<sup>2</sup> (two discrete parameters)
- wing veins: *Q* = ℜ<sup>2</sup> (two continuous parameters)

**Motivation.** Topological space *X* filtered by set *Q* of subspaces: *X*<sub>*q*<sub> ⊆ *X* for *q* ∈ *Q* ⇒ partial ordering on *Q* via *X*<sub>*q*<sub> ⊆ *X*<sub>*q′*<sub> ⇔ *q* ≤ *q′*

**Def.** {*X*<sub>*q*<sub>}<sub>*q*∈*Q* has persistent homology the *Q*-module *H* with *H*<sub>*q* = *H*(*X*<sub>*q*<sub>; *k*)
**Modules over posets**

**Def.** *Q*-module over the poset *Q*: (think *Q* = \(\mathbb{Z}^n\) or \(\mathbb{R}^n\))

- *Q*-graded vector space \(M = \bigoplus_{q \in Q} M_q\) over the field \(k\) with
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**Examples**

- \(Q = \mathbb{Z}^n \Leftrightarrow M = \mathbb{Z}^n\)-graded \(k[x_1, \ldots, x_n]\)-module
- \(Q = \mathbb{R}^n \Leftrightarrow M = \mathbb{R}^n\)-graded \(k[\mathbb{R}^n_+]\)-module
- wing veins: \(Q = \mathbb{Z}^2\) (two discrete parameters)
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**Def.** \(\{X_q\}_{q \in Q}\) has persistent homology the *Q*-module \(H\) with \(H_q = H(X_q; k)\)
Modules over posets

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Modules over posets

Def. **Q-module** over the poset $Q$: (think $Q = \mathbb{Z}^n$ or $\mathbb{R}^n$)

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Def. Q-module over the poset Q: (think $Q = \mathbb{Z}^n$ or $\mathbb{R}^n$)

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Def. $\{X_q\}_{q \in Q}$ has persistent homology the Q-module $H$ with $H_q = H(X_q; k)$
Example: planar maps [with Houle, et al. (many advisees)]

Use two parameters to encode fruit fly wing

- **1st parameter**: distance from vertex set
- **2nd parameter**: distance from edge set

Sublevel set $W_{r,s}$ is near edges but far from vertices

$\mathbb{Z}^2$-module:
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$\mathbb{Z}^2$-module:

$$
\begin{align*}
&\rightarrow H_{r-\epsilon, s+\delta} \rightarrow H_{r, s+\delta} \rightarrow H_{r+\epsilon, s+\delta} \rightarrow \\
&\rightarrow H_{r-\epsilon, s} \rightarrow H_{r, s} \rightarrow H_{r+\epsilon, s} \rightarrow \\
&\rightarrow H_{r-\epsilon, s-\delta} \rightarrow H_{r, s-\delta} \rightarrow H_{r+\epsilon, s-\delta} \rightarrow
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$\mathbb{Z}^2$-module:
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- **1st parameter**: distance from vertex set (require distance $\geq -r$)
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- **1st parameter**: distance from vertex set (require distance $\geq -r$)
- **2nd parameter**: distance from edge set (require distance $\leq s$)

Sublevel set $W_{r,s}$ is near edges but far from vertices

$\mathbb{Z}^2$-module:

$\begin{align*}
\rightarrow H_{r-\varepsilon,s+\delta} &\rightarrow H_{r,s+\delta} &\rightarrow H_{r+\varepsilon,s+\delta} \\
\uparrow &\uparrow &\uparrow \\
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- 2nd parameter: distance from edge set (require distance $\leq s$)

Sublevel set $W_{r,s}$ is near edges but far from vertices

$\mathbb{Z}^2$-module:

$$
\begin{align*}
&\rightarrow H_{r-\epsilon,s+\delta} \rightarrow H_{r,s+\delta} \rightarrow H_{r+\epsilon,s+\delta} \rightarrow \\
&\uparrow \hspace{2cm} \uparrow \hspace{2cm} \uparrow \hspace{2cm} \uparrow \\
&\rightarrow H_{r-\epsilon,s} \rightarrow H_{r,s} \rightarrow H_{r+\epsilon,s} \rightarrow \\
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Example: planar maps [with Houle, et al. (many advisees)]

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Modules over posets

Def. **Q-module** over the poset **Q**: (think **Q** = **Z**^n or **R**^n)

- **Q**-graded vector space **M** = \( \bigoplus_{q \in Q} M_q \) over the field \( k \) with
- homomorphism \( M_q \to M_{q'} \) whenever \( q \preceq q' \) in **Q** such that
- \( M_q \to M_{q''} \) equals the composite \( M_q \to M_{q'} \to M_{q''} \) whenever \( q \preceq q' \preceq q'' \)

Examples

- \( Q = \mathbb{Z}^n \iff M = \mathbb{Z}^n\)-graded \( k[x_1, \ldots, x_n] \)-module
- \( Q = \mathbb{R}^n \iff M = \mathbb{R}^n\)-graded \( k[\mathbb{R}^n_+] \)-module
- wing veins: \( Q = \mathbb{Z}^2 \) (two discrete parameters)
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Motivation. Topological space \( X \) filtered by set \( Q \) of subspaces:
\( X_q \subseteq X \) for \( q \in Q \Rightarrow \) partial ordering on \( Q \) via \( X_q \subseteq X_{q'} \iff q \preceq q' \)

Def. \( \{X_q\}_{q \in Q} \) has **persistent homology** the \( Q \)-module \( H \) with \( H_q = H(X_q; k) \)
**Modules over posets**

**Def.** *Q*-module over the poset *Q*: (think *Q* = \( \mathbb{Z}^n \) or \( \mathbb{R}^n \))

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Partially ordered groups

Monomial ideals: combinatorial primary decomposition
- teases apart groups of monomials parallel to the coordinate planes
- or parallel to faces in affine semigroup rings

Combinatorial primary decomposition requires positive multigrading
⇔ cone of positive elements greater than 0.

Question. Over which posets does monomial primary decomposition work?

Remark. Homological algebra generalizes to modules over any poset

Answer 1 [–]. Need at least
- faces (⇔ prime ideals)
- localization (without altering ambient poset)

Def. Abelian group $Q$ is partially ordered if $Q$ generated by submonoid $Q_+$, the positive cone, with trivial unit group. Partial order: $q \preceq q' \iff q' - q \in Q_+$

Lemma. $Q$-module $\iff Q$-graded module over $k[Q_+]$
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Examples.

1. $Q = \mathbb{Z}^n$ with $Q_+ = \mathbb{N}^n$: polynomial combinatorial commutative algebra
2. $Q = \mathbb{Z}^n$ with $Q_+$ pointed rational polyhedral cone: affine semigroup CCA
3. $Q = \mathbb{R}^n$ with $Q_+$ any pointed polyhedral cone: the case of interest here
   - rational or irrational okay, but need
   - finitely many faces
4. $Q = \mathbb{R}^3$ with $Q_+$ = positive half of the right circular cone $x^2 + y^2 \leq z^2$
5. $Q = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $Q_+$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
6. continuous version: $Q = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ with $Q_+$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7. $\mathbb{R}^n \hookrightarrow \mathbb{Q}^n$
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• line = \( \mathbb{R} \)
• origin \( \mathbf{0} \) at dot
• \( Q_+ \) shaded

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Tameness

Finiteness conditions: ● $\mathbb{Z}^n$-modules: finitely generated $\Leftrightarrow$ noetherian
● $Q =$ partially ordered group: ??

Def [−]. $M$ admits a constant subdivision if $Q$ is partitioned into
● constant regions $A \rightsquigarrow$ vector space $M_A \simrightarrow M_a$ for all $a \in A$ with
● no monodromy: all comparable pairs $a \preceq b$ with $a \in A$ and $b \in B$ induce
the same composite $M_A \rightarrow M_a \rightarrow M_b \rightarrow M_B$.

$M$ is tame if $\dim_k M_q < \infty$ and admits a finite constant subdivision.
Example. $k_0 \oplus k[\mathbb{R}^2]$ admits constant regions $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$
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**Tameness**

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- constant regions \( A \rightsquigarrow \) vector space \( M_A \sim \to M_a \) for all \( a \in A \) with
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\( M \) is tame if \( \text{dim}_k M_q < \infty \) and admits a finite constant subdivision.

**Example.** \( k_0 \oplus k[R^2] \) admits constant regions \( \{0\} \) and \( \mathbb{R}^2 \setminus \{0\} \)
Tameness

Finiteness conditions: • \( \mathbb{Z}^n \)-modules: finitely generated \( \Leftrightarrow \) noetherian
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Def. Fix a partially ordered group $Q$. A **face** of $Q_+$ (or of $Q$ itself) is
- a submonoid $\sigma \subseteq Q_+$ such that
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Answer 2 [–]. $Q$ is **polyhedral** if it has only finitely many faces.

Thm [–]. Fix a polyhedral partially ordered group $Q$. Any tame $Q$-module $M$ has a **primary decomposition**: $M \hookrightarrow \bigoplus \limits_{\text{faces } \tau} M_{\tau}$ with $M_{\tau}$ coprimary.

Remark. $0 = \bigcap_{\tau} \ker(M \to M_{\tau})$ with $\ker(M \to M_{\tau})$ primary in $M$

Example.

Thm [–]. Canonical for monomial quotients $M$
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Example.

$$
\begin{bmatrix}
  y \\
v \\
x
\end{bmatrix}
\hookrightarrow
\begin{bmatrix}
y \\
v \\
x
\end{bmatrix}
\oplus
\begin{bmatrix}
y \\
v \\
x
\end{bmatrix}
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y \\
v \\
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Remark. Minimality needs $\mathbb{Z}^n$ or $\mathbb{R}^n$ [arXiv:2008.03819] or other geometric control
Coprimary modules

Def [–]. In a partially ordered group $Q$, a ray of $Q_+$ (or of $Q$ itself) is a face that is totally ordered as a partially ordered submonoid of $Q$.

Answer 3 [–]. $Q$ is closed if the complement $Q_+ \setminus \tau$ of each face $\tau$ is generated as an ideal of $Q_+$ by $\rho \setminus \{0\}$ for the rays $\rho \nsubseteq \tau$.

Def [–]. If $Q$ is closed and $\tau$ is a face, then a $Q$-module element is
1. $\tau$-persistent if it lives when pushed up arbitrarily along $\tau$;
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A $Q$-module is $\tau$-coprimary if every element divides a coprimary element.

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\[ x y \cdots \mapsto \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
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**Computation.** Given a tame module, compute

- primary decomposition
- indecomposable decomposition ← in progress with undergrad Joey Li

**Betti numbers.**

- analogue of Hochster’s formula for real-exponent monomial ideals
- Does \( k[\mathbb{R}^n_+] \) have finite global dimension in the usual sense?
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**Bar codes / QR codes**

- descriptions of modules in terms of “birth” and “death” (generators and cogenerators)
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