The tropical Grassmannian $\mathrm{TGr}_0(3, 8)$, Dressian $\mathrm{Dr}(3, 8)$, and their positive parts

Benjamin Schröter

KTH Royal Institute of Technology

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The Grassmannian

The Grassmannian $\mathrm{Gr}(r, \mathbb{K}^n)$ is the algebraic variety generated by

$$\sum_{j \in J \setminus I} (-1)^{\sigma(j,I,J)} \, p_{J \setminus j} \, p_{I \cup j} \qquad \text{where } I \in \binom{[n]}{r-1}, J \in \binom{[n]}{r+1}.$$

It is the moduli space of all r-dimensional subspaces of $\mathbb{K}^n$.
Most of the time we assume that $\mathbb{K} = \mathbb{C}\{\!\{t\}\!\}$ is the algebraically closed field of Puiseux series over $\mathbb{C}$ equipped with a valuation $\mathrm{val}(z) = 0$ for $z \in \mathbb{C}$ and $\mathrm{val}(t) = 1$.

Example

As projective varieties $\mathrm{Gr}(1, \mathbb{C}^n)$ and $\mathrm{Gr}(n-1, n)$ are $\mathbb{P}^{n-1}_{\mathbb{C}}$ and $\mathrm{Gr}(2, \mathbb{C}\{\!\{t\}\!\}^4)$ consists of all vectors $(w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}) \in \mathbb{P}^5_{\mathbb{C}\{\!\{t\}\!\}}$ with

$$w_{12} w_{34} - w_{13} w_{24} + w_{14} w_{23} = 0$$

e.g. $w = (1, 1, 1, 1-t, 1, t)$.

Initial forms and initial ideals

Given a vector $\omega \in \mathbb{R}^n$ and a polynomial $f = \sum_u c_u \cdot x^u \in \mathbb{C}\{\{t\}\}[x_1, \ldots .x_n]$.

$$\mathrm{in}_\omega(f) = \sum_{\substack{\mathrm{val}(c_u)+u\cdot\omega \\ \text{minimal}}} \overline{c^u \cdot t^{-\mathrm{val}(c_u)}} \cdot x^u \in \mathbb{C}[x_1, \ldots, x_n]$$

here $\overline{\cdot}$ is the evaluation at $t = 0$ that projects onto the residue field $\mathbb{C}$.
The initial ideal of an ideal $\mathcal{I}$ with respect to ω is given by

$$\mathrm{in}_\omega(\mathcal{I}) = \{\, \mathrm{in}_\omega(f) \mid f \in \mathcal{I}\}$$

Example

$$\begin{aligned} \mathcal{I}_{2,4} &= \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle \\ \mathrm{in}_{e_{34}}(\mathcal{I}_{2,4}) &= \langle p_{13}p_{24} - p_{14}p_{23} \rangle \\ \mathrm{in}_{-e_{34}}(\mathcal{I}_{2,4}) &= \langle p_{12}p_{34} \rangle \end{aligned}$$

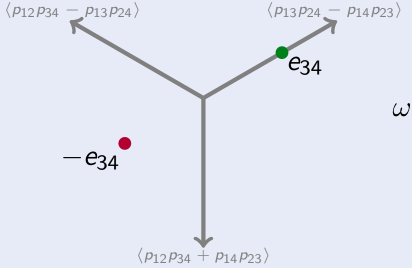
Tropical varieties

The tropical variety of an ideal $\mathcal{I} \trianglelefteq \mathbb{K}[x_1, \ldots, x_m]$ is the set

$$\mathrm{Trop}(\mathcal{I}) = \{\omega \in \mathbb{R}^m \mid \mathrm{in}_\omega(\mathcal{I}) \text{ is monomial free}\}$$

The condition $\mathrm{in}_\omega(\mathcal{I}) = \mathrm{in}_\nu(\mathcal{I})$ defines a equivalence relation on $\mathbb{R}^n$, which imposes the Gröbner structure on $\mathrm{Trop}(\mathcal{I})$.
The tropical Grassmannian is $\mathrm{TGr}_p(r, n) = \{\mathrm{val}(w) \mid w \in \mathrm{Gr}(r, \mathbb{K}^n)\}$ which depends on the characteristic p of $\mathbb{K}$.

Example



$$\mathrm{in}_\omega(\mathcal{I}_{2,4}) = \langle p_{13}p_{24} - p_{14}p_{23}\rangle$$
is equivalent to
$$\omega_{12} + \omega_{34} > \omega_{13} + \omega_{24} = \omega_{14} + \omega_{23}$$

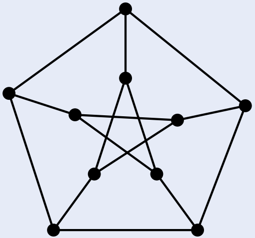
The tropical Grassmannian $\mathrm{TGr}_p(2, n)$

Theorem (Speyer, Sturmfels 2004)

The tropical Grassmannian $\mathrm{TGr}_p(2, n)$ is independent of the characteristic. Equipped with the Gröbner structure it becomes a simplicial fan which has $2^{n-1} - n - 1$ rays and $(2n-5)!!$ maximal cones. This is the coarsest fan structure on $\mathrm{TGr}_p(2, n)$. It is the moduli spaces of tropical lines in $\mathbb{R}^n$, phylogenetic trees with n labeled leaves, and tropical rational curves of genus 0 with n marked points.

Example

$TGr_p(2, 5)$ has $2^4 - 6 = 10$ rays and $5!! = 5 \cdot 3 \cdot 1 = 15$ maximal cones. Intersected with a sphere it is an embedding of the Peterson graph.

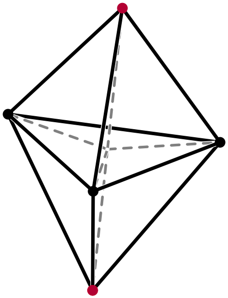


The tropical Grassmannian $\mathrm{TGr}_p(3,6)$

Theorem (Speyer, Sturmfels 2004)

The tropical Grassmannian $\mathrm{TGr}_p(3,6)$ *is independent of the characteristic. Equipped with the Gröbner structure it is a simplicial fan with* 65 *rays and* 1035 *maximal cones. There is a coarsening that consists of* 1005 *maximal cones.*

Tropical varieties are computed via a traverse of a Gröbner complex. Every cone requires a Gröbner basis computation.



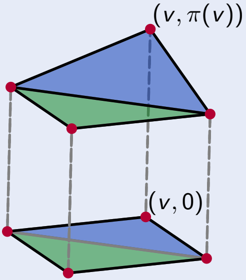
The Dressian

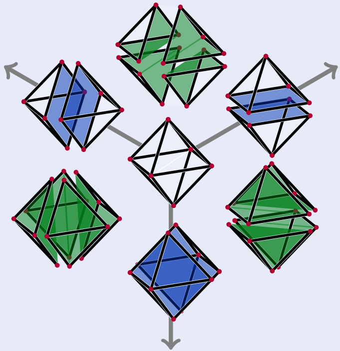
The moment polytope of the Grassmannian is the hypersimplex

$$\Delta(r,n) \;=\; \left\{ x \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i = r \right\} \;.$$

The regular subdivisions of $\Delta(r,n)$ that do not introduce new edges form the Dressian $\mathrm{Dr}(r,n)$ as a subfan of the secondary fan of $\Delta(r,n)$.

Example





The structure of the Dressian

Theorem (Olarte, Panizzut, S. 2019)

The Dressian $Dr(r,n)$ with the secondary fan structure agrees with

$$\bigcap_{I,a,b,c,d} \mathrm{Trop}(\langle p_{Iab}p_{Icd} - p_{Iac}p_{Ibd} + p_{Iad}p_{Ibc}\rangle)$$

where $I \in \binom{[n]}{r-2}$ and $a,b,c,d \notin I$. This structure is coarse.

The tropical Grassmannian $\mathrm{TGr}_p(r,n)$ is a subset of the Dressian $\mathrm{Dr}(r,n)$. Dressians are studied as moduli space of valuated matroids, and projections as gross substitutes in economics.

The tropical Grassmannian $\mathrm{TGr}_p(3,7)$

Theorem (Herrmann, Jensen, Joswig, Sturmfels 2009)

The tropical Grassmannians $\mathrm{TGr}_0(3,7)$ *and* $\mathrm{TGr}_2(3,7)$ *are different sets. The former consists of* 751 *rays and* 252 000 *maximal cones as a subfan of the Gröbner fan. The coarsest structure on* $\mathrm{TGr}_0(3,7)$ *is those as a subfan of the secondary fan of* $\Delta(3,7)$ *with* 211 365 *maximal cones. The tropical Grassmannian* $\mathrm{TGr}_2(3,7)$ *is not a subfan of this secondary fan.*

There is a single S_7-orbit of a polynomial that whose tropical hypersurfaces decide which elements of $\mathrm{Dr}(3,7)$ belong to $\mathrm{TGr}_0(3,7)$.

$$\begin{aligned} f = 2p_{123}p_{467}p_{567} - p_{367}p_{567}p_{124} - p_{167}p_{467}p_{235} \\ -p_{127}p_{567}p_{346} - p_{126}p_{367}p_{457} - p_{237}p_{467}p_{156} \\ +p_{134}p_{567}p_{267} + p_{246}p_{567}p_{137} + p_{136}p_{267}p_{457} \in \mathcal{I}_{3,7} \end{aligned}$$

The tropical Grassmannian $\mathrm{TGr}_0(3,8)$

Theorem (Bendle, Böhm, Ren, S. 2020^+)

The Gröbner subfan supported on the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ *is a* 16*-dimensional fan it consists of* 732 725 *rays and* 278 576 760 *maximal cones. Moreover, the coarsest fan structure supported on* $\mathrm{TGr}_0(3,8)$ *is a subfan of the secondary fan of* $\Delta(3,8)$ *which consists of* 15 470 *rays and* 117 445 125 *maximal cones.*
In particular, there is a maximal cone that gets refined into 2 620 *maximal cones by the Gröbner structure.*

The Dressian is a 17-dimensional non-pure fan with 117 595 485 cones of dimension 16. The 3-term Plücker relations and the S_8-orbit of the polynomial $f \in \mathcal{I}_{3,7} \subset \mathcal{I}_{3,8}$ form a tropical basis for $\mathrm{TGr}_0(3,8)$.

The structure of $\mathrm{TGr}_0(r, n)$

The saturation of $\mathcal{I}$ is the ideal

$$\{ f \mid \text{There is a monomial } x^m \text{ such that } x^m f \in \mathcal{I} \}$$

Conjecture

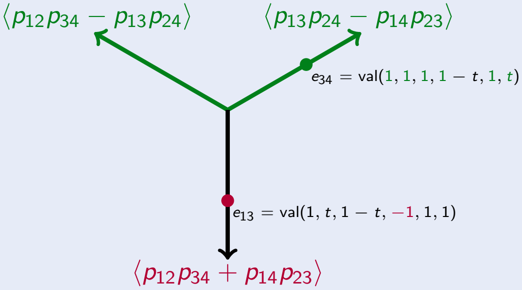
The tropical Grassmannian $\mathrm{TGr}_0(r, n)$ *is a subfan of the secondary fan of* $\Delta(r, n)$*. Moreover, for* $\nu, \omega \in \mathrm{TGr}_0(r, n)$ *we have* ν *and* ω *lie in the same relative open cone if and only if the two ideals* $\mathrm{in}_\nu(\mathcal{I}_{r,n})$ *and* $\mathrm{in}_\omega(\mathcal{I}_{r,n})$ *agree after saturation.*

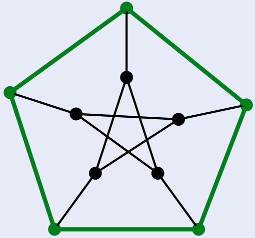
The positive part of $\mathrm{TGr}_0(r, n)$

The positive part of a tropical variety is

$$\mathrm{Trop}^+(\mathcal{I}) = \{\omega \in \mathrm{Trop}(\mathcal{I}) \mid \mathrm{in}_\omega(I) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle\}$$

Example





The positive tropical Grassmannian $\mathrm{TGr}^+(3,8)$

We developed algorithms that identify top dimensional positive cones and positivity of binomial (initial) ideals. We could verify computational the following for $\mathrm{TGr}^+(3,8)$.

Theorem (Speyer, Williams 2020, Arkani-Hamed, Lam, Spradlin 2020)

The positive tropical Grassmannian $\mathrm{TGr}^+(r,n)$ is a subfan of the secondary fan of $\Delta(r,n)$.

Theorem (Brodsky, Stump 2018)

With the above structure, $\mathrm{TGr}^+(3,8)$ is a pure 16-dimensional fan with 120 rays and 13 612 maximal cones. There exist a refinement into simplicial cones such that $\mathrm{TGr}^+(3,8)$ is isomorphic to the cluster complex of Coxeter type E_8

Scattering amplitudes and $\mathrm{TGr}^+(r, n)$

The maximally supersymmetric Yang-Mills theory in four dimensions is a model in theoretical physics that describes tree-level scattering amplitudes when finitely many massless particles interact.

- Plücker relations appear as Jacobi and Schouten identity.
- Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov and Trnka (2016) related Feyman digrams or more generally on-shell diagrams to the stratification of positive Grassmannians.
- Cachazo, Early, Guevara and Mizeraa (2019) studied biadjoint amplitudes in terms of Mandelstam and Kinematic variables in terms of rays and maximal cones of $\mathrm{TGr}^+(3, n)$.