

Real algebraic geometry reading group topics

MPI Leipzig July 3–7, 2017.

The following are proposed topics for talks to be delivered by the participants of the workshop. Further suggestions from you the participants are also welcome!

Topology of real algebraic varieties

- **Constructing real algebraic varieties with interesting topology**

This talk should briefly describe the construction methods of Harnack, Hilbert, and Gudkov [Utk78] and an explanation of the classification of planar sextics. Then give a description of Viro’s patchworking construction and theorem [Vir01] from the point of view of tropical geometry [IMS09]. This could also include Haas’ patchworking theorem for curves [BIMS15, Chapter 3.3].

- **Topology of real K3 surfaces**

This talk should describe Nikulin’s classification of real K3 surfaces and planar sextics up to rigid isotopy via lattice theory [Nik80]. If time permits the speaker could also mention the classification of 1-nodal planar sextics by Itenberg by similar methods [Ite94].

- **Topology of real algebraic space curves**

While the topology of planar curves has been extensively studied, the study of the topology of real algebraic curves in \mathbb{RP}^3 remained relatively unexamined. This talk should address the rigid isotopy classification of smooth irreducible algebraic curves in the real projective 3-space for small degree and genus [Bj11, MO16] and of canonically embedded curves of genus four [DZ99].

Convex algebraic geometry

- **Spectrahedral shadows**

Spectrahedral shadows are the feasible sets of semidefinite programming and thus the question of their characterisation is of high interest. A large class of semi-algebraic convex sets are in fact spectrahedral shadows as shown in [HN09, HN10]. These results lead to the Helton–Nie Conjecture stating that in fact every semi-algebraic convex set is a spectrahedral shadow and could be the content of the first talk.

- **Convex hulls of curves**

This conjecture was proved to be correct in the case of two dimensional convex sets in [Sch12]. More generally it was shown that the convex hull of a semialgebraic set of dimension one is always a spectrahedral shadow. This could be the content of another talk.

- **Counter-examples to the Helton–Nie conjecture**

The last talk about spectrahedral shadows could address a very recent family of counter-examples to the Helton–Nie Conjecture [Sch16].

- **Spectrahedra**

Plane Spectrahedra (and even the size of their describing matrices) are characterised by [HV07] by a total reality condition. The old proof is heavily based on results from [Vin93] and explicitly gives a description as a spectrahedron in terms of theta functions. There is a remarkably easier proof only using basic real algebra and the 2-divisibility of the Jacobian of a curve [Han16]. This talk should give a brief description (without details) of the original proof and a presentation of the new argument.

Random real algebraic geometry

- **Topology of random real algebraic varieties**

This talk will present results on the expected number of connected components of a random real algebraic hypersurface in projective space from [FLL15]. More generally, the results on the expected Betti numbers of real algebraic varieties may also be surveyed [Wel14], [GW14].

- **Random real Schubert calculus and lines on surfaces**

We propose one talk to be on the expected number of lines on random real and complex algebraic hypersurfaces [BLLP16]. Another another talk can describe the asymptotic results of the number of real solutions to Schubert problems from [BL16].

- **Random real tensors**

Another possible direction would be properties of random real tensors as considered for example in [DH16].

References

- [BIMS15] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw. Brief introduction to tropical geometry. *arXiv preprint arXiv:1502.05950*, 2015.
- [Bj11] Johan Björklund. Real algebraic knots of low degree. *J. Knot Theory Ramifications*, 20(9):1285–1309, 2011.

- [BL16] Peter Bürgisser and Antonio Lerario. Probabilistic schubert calculus. *arXiv preprint arXiv:1612.06893*, 2016.
- [BLLP16] Saugata Basu, Antonio Lerario, Erik Lundberg, and Chris Peterson. Random fields and the enumerative geometry of lines on real and complex hypersurfaces. *arXiv preprint arXiv:1610.01205*, 2016.
- [DH16] Jan Draisma and Emil Horobeț. The average number of critical rank-one approximations to a tensor. *Linear and Multilinear Algebra*, 64(12):2498–2518, 2016.
- [DZ99] A. I. Degtyarev and V. I. Zvonilov. Rigid isotopy classification of real algebraic curves of bidegree $(3, 3)$ on quadrics. *Mat. Zametki*, 66(6):810–815, 1999.
- [FLL15] Yan V Fyodorov, Antonio Lerario, and Erik Lundberg. On the number of connected components of random algebraic hypersurfaces. *Journal of Geometry and Physics*, 95:1–20, 2015.
- [GW14] Damien Gayet and Jean-Yves Welschinger. Expected topology of random real algebraic submanifolds. <https://arxiv.org/pdf/1307.5287.pdf>, 2014.
- [Han16] Christoph Hanselka. Characteristic polynomials of symmetric matrices over the univariate polynomial ring. *arXiv preprint arXiv:1610.06634*, 2016.
- [HN09] J. William Helton and Jiawang Nie. Sufficient and necessary conditions for semidefinite representability of convex hulls and sets. *SIAM J. Optim.*, 20(2):759–791, 2009.
- [HN10] J. William Helton and Jiawang Nie. Semidefinite representation of convex sets. *Math. Program.*, 122(1, Ser. A):21–64, 2010.
- [HV07] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5):654–674, 2007.
- [IMS09] Ilia Itenberg, Grigory Mikhalkin, and Eugenio I Shustin. *Tropical algebraic geometry*, volume 35. Springer Science & Business Media, 2009.
- [Ite94] IV Itenberg. Rigid isotopy classification of curves of degree 6 with one nondegenerate double point. *Topology of manifolds and varieties (ed. O. Viro), Advances in Soviet Mathematics*, 18, 1994.
- [MO16] Grigory Mikhalkin and Stepan Orevkov. Real algebraic knots and links of small degree. *J. Knot Theory Ramifications*, 25(12):1642010, 34, 2016.
- [Nik80] Vyacheslav Valentinovich Nikulin. Integral symmetric bilinear forms and some of their applications. *Izvestiya: Mathematics*, 14(1):103–167, 1980.
- [Sch12] Claus Scheiderer. Semidefinite representation for convex hulls of real algebraic curves. *arXiv preprint arXiv:1208.3865*, 2012.

- [Sch16] Claus Scheiderer. Semidefinitely representable convex sets. *arXiv preprint arXiv:1612.07048*, 2016.
- [Utk78] Dmitri Andreevich Gudkov GA Utkin. *Nine Papers on Hilbert's 16th Problem*. American Mathematical Soc., 1978.
- [Vin93] Victor Vinnikov. Selfadjoint determinantal representations of real plane curves. *Math. Ann.*, 296(3):453–479, 1993.
- [Vir01] Oleg Viro. Dequantization of real algebraic geometry on logarithmic paper. In *European Congress of Mathematics*, pages 135–146. Springer, 2001.
- [Wel14] Jean-Yves Welschinger. Topology of random real hypersurfaces. <http://math.univ-lyon1.fr/welschinger/CIMPA.pdf>, 2014.