LET'S GET REAL: WRAP UP AND COMPUTATIONS
EXERCISE 14

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Exercise 14: Consider the preordering

$$T = \{f \in \mathbb{K}[x,y] \mid f \geq 0 \text{ on regular hexagon}\}$$

of all polynomials in two variables that are nonnegative on a regular hexagon. Is it finitely generated? Stable? Archimedian? Does the choice of field (like $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{R}$, or $\mathbb{R}\{\{\varepsilon\}\}$) matter?

The following solution is based on the book of Marshall [Ma].

Definition 1. *Let* A *be a commutative ring with* 1 *(*2 *is not a unit). A* preordering *of* A *is a subset* T *of* A *such that*

i) $T + T \subseteq T$,
ii) $T \cdot T \subseteq T$, *and*
iii) $a^2 \in T$ *for all* $a \in \mathsf{A}$.

Definition 2. *Let* $S = \{g_1, ..., g_s\} \subset \mathsf{A}$ *be a finite subset of* A*. The* preodering T_S generated by S *is defined by*

$$T_S := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e g^e \,\middle|\, \sigma_e \in \sum \mathsf{A}^2 \right\}, \tag{1}$$

where $g^e = g_1^{e_1} \cdots g_s^{e_s}$*. A preordering* T *is called* finitely generated *if a finite* $S \subset \mathsf{A}$ *exists with* $T = T_S$*.*

Definition 3. *Let* $S = \{g_1, ..., g_s\} \subset \mathsf{A}$ *be finite. Define* K_S *as*

$$K_S := \{x \in \mathbb{K}^n \mid g_i(x) \geq 0 \text{ for all } i = 1, ..., s\}.$$

At first, let us express the regular hexagon in terms of K_S.

Proposition 4. *The regular hexagon with vertices* $\{(\pm 1, 0), (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\}$ *is generated by the set* $S = \{g_1, ..., g_6\}$ *with*

$$g_1 := -\frac{\sqrt{3}x}{2} - \frac{y}{2} + \frac{\sqrt{3}}{2} \qquad g_2 := -y + \frac{\sqrt{3}}{2}$$
$$g_3 := \frac{\sqrt{3}x}{2} - \frac{y}{2} + \frac{\sqrt{3}}{2} \qquad g_4 := \frac{\sqrt{3}x}{2} + \frac{y}{2} + \frac{\sqrt{3}}{2}$$
$$g_5 := y + \frac{\sqrt{3}}{2} \qquad g_6 := -\frac{\sqrt{3}x}{2} + \frac{y}{2} + \frac{\sqrt{3}}{2},$$

i.e., $H = K_S$*.*

So the question, whether T in the exercise is finitely generated or not is translated to whether T_S is saturated or not due to the following.

Definition 5. *Let* T *be a preordering and* $\operatorname{Sper}_T(\mathsf{A})$ *be the set of all orderings* P *of* A *satisfying* $T \subseteq P$*. Then we define the* saturation $\tilde{T}$ *of* T *as*

$$\tilde{T} := \bigcap_{P \in \operatorname{Sper}_T(\mathsf{A})} P.$$

Lemma 6. *Let S be a finite subset of $\mathbb{R}[\underline{X}]$. Then*

$$\tilde{T}_S = \{f \in \mathbb{R}[\underline{X}] \mid f \geq 0 \text{ on } K_S\}.$$

So by the previous lemma we have $T = \tilde{T}_S$. T_S is saturated iff it is finitely generated. From the following fact we already know that T_S is quite large.

Theorem 7 (Schmüdgen)**.** *$f > 0$ on K_S, then $f \in T_S$.*

So T_S already contains all $f > 0$ on K_S. But in general there are $f \geq 0$ on K_S with $f \notin T_S$. In dimension ≥ 3 this always happens, but in dimension 2 both cases appear, depending on K_S.

Theorem 8 (Scheiderer)**.** *Suppose K_S is a compact set in $\mathbb{R}^2$ defined by polynomial inequalities $g_1 \geq 0$, ..., $g_s \geq 0$, each g_i is irreducible in $\mathbb{R}[X,Y]$ and, for each boundary point p of K_S, either*

(1) $\exists i$ such that $g_i(p) = 0$, p is a non-singular zero of g_i, and K_S is defined in a neighborhood of p by the single inequality $g_i \geq 0$; or

(2) $\exists i, j$ such that g_i and g_j meet transversally at p, and K_S is defined in a neighborhood of p by $g_i \geq 0$ and $g_j \geq 0$.

Then the preorder T_S in $\mathbb{R}[X,Y]$ is saturated.

So by the previous theorem T is finitely generated over $\mathbb{K} = \mathbb{R}$. For $\mathbb{K} = \mathbb{Q}$ we still can find $H = K_{S'}$ with $S' \subset \mathbb{Q}[X,Y]$, e.g., $S' = \{f_1, ..., f_4\}$ with

$$f_1 := g_1 g_6 = \frac{3x^2}{4} - \frac{3x}{4} - \frac{y^2}{4} + \frac{3}{4} \qquad f_2 := g_2 g_5 = -y + \frac{3}{4}$$
$$f_3 := g_3 g_4 = \frac{3x^2}{4} + \frac{3x}{2} - \frac{y^2}{4} + \frac{3}{4} \qquad f_4 := -x^2 - y^2 + 1,$$

but Scheiderer's Theorem no longer applies for $\mathbb{Q}$. In fact, there are polynomials in $\mathbb{Q}[X,Y]$ which are sums of squares over $\mathbb{R}$ but not over $\mathbb{Q}$. Additionally, the set S' no longer fulfills Scheiderers conditions (1) and (2).

For the real closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ it seems reasonable that T_S is still saturated. Since $\sqrt{3} \in \overline{\mathbb{Q}}$ we have $g_i \in \overline{\mathbb{Q}}[X,Y]$, and if $f \in \overline{\mathbb{Q}}[X,Y]$ with $f = \sum_{e \in \{0,1\}^s} g^e \sigma_e$ in $\mathbb{R}[X,Y]$, then one can even choose $\sigma_e \in \overline{\mathbb{Q}}[X,Y]$ by the Tarski principle.

For $\mathbb{K} = \mathbb{R}\{\{\varepsilon\}\}$ it seems to be an open problem.

Definition 9. *A preordering T is called* Archimedian *if for any $a \in \mathsf{A}$ there is a $n = n(a) \in \mathbb{N}$ such that $a + n \in T$.*

Theorem 10 (Wörmann)**.** *T_S is Archimedean iff K_S is compact.*

So by the previous theorem we have that T_S (and also T) is Archimedean for $\mathbb{K} = \mathbb{Q}, \overline{\mathbb{Q}}, \mathbb{R}$. For $\mathbb{K} = \mathbb{R}\{\{\varepsilon\}\}$ it is not immediately clear since $\mathbb{R}\{\{\varepsilon\}\}$ is not Archimedean, but this is not enough that it fails for T_S. So it seems to be still open.

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References

[Ma] M. Marshall, *Positive Polynomials and Sums of Squares*, American Mathematical Society, Rhode Island, 2008.