Algebraic Combinatorics in Geometric Complexity Theory

Greta Panova

University of Southern California

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Combinatorics and Representation Theory basics

**Symmetric group** $S_n$: Permutations $\pi : [1..n] \mapsto [1..n]$ under composition.

**Integer partitions** $\lambda \vdash n$:

$$\lambda = (\lambda_1, \ldots, \lambda_\ell), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0, \lambda_1 + \lambda_2 + \cdots = n$$

**Young diagram** of $\lambda$: 

Here $\lambda = (5, 3, 2)$
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Representations of $S_n$: group homomorphisms $S_n \rightarrow GL(V)$,

Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi(i)}$ for $i = 1..3$, so e.g. $231 \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
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**Irreducible decomposition**: minimal $S_n$-invariant subspaces $V_i$, so $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, e.g. $V = \mathbb{C}\langle e_1 + e_2 + e_3 \rangle \oplus \mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle$

The **irreducible modules (representations)** (up to equivariant isomorphisms) of $S_n$ are the **Specht modules** $S_\lambda$, indexed by all $\lambda \vdash n$, e.g. $V_1 \simeq S_\begin{array}{ll} 3 \\ 2 \end{array}$ and $V_2 \simeq S_\begin{array}{ll} 3 \\ 1 \end{array}$
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The irreducible modules (representations) (up to equivariant isomorphisms) of $S_n$ are the Specht modules $S_\lambda$, indexed by all $\lambda \vdash n$, e.g. $V_1 \simeq S_\begin{array}{ccc} 1 \\ 2 \end{array}$ and $V_2 \simeq S_\begin{array}{cc} 2 \\ 4 \end{array}$

Basis for $S_\lambda$: Standard Young Tableaux of shape $\lambda$: $\lambda = (3, 2)$

1 2 3 4 5
1 2 4 3 5
1 2 5 3 4
1 3 4 2 5
1 3 5 2 4
Young Tableaux and Schur functions

*Irreducible representations* of the symmetric group $S_n$: *Specht modules* $S_\lambda$

![Young Tableaux](image)

*Irreducible (polynomial) representations* of the *General Linear group* $GL_N(\mathbb{C})$:

Weyl modules $V_\lambda$ (aka $W_\lambda$), indexed by highest weights $\lambda$, $\ell(\lambda) \leq N$. 
Young Tableaux and Schur functions

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\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 2 & 5 \\
4 & 5 & 2 & 4 \\
\end{array}
\]

Irreducible (polynomial) representations of the General Linear group $GL_N(\mathbb{C})$:

Weyl modules $V_\lambda$ (aka $W_\lambda$), indexed by highest weights $\lambda$, $\ell(\lambda) \leq N$.

Schur functions: characters of $V_\lambda$

$Tr_{V_\lambda}(\text{diag}(x_1, \ldots, x_N)) = s_\lambda(x_1, \ldots, x_N)$

Weyl’s determinantal formula:

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[ x_i^{\lambda_j + N - j} \right]_{ij=1}^N}{\prod_{i<j}(x_i - x_j)}
\]

Semi-Standard Young tableaux of shape $\lambda$:

$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3^2 + x_1 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$.

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 1 & 3 & 3 \\
2 & 2 & 3 & 3 \\
1 & 2 & 2 & 3 \\
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\end{array}
\]
Products and compositions

Von Neumann et al, ca. 1934, representations of Lie groups:

\( GL_N(\mathbb{C}) \) acts on \( V_\lambda, V_\mu \) and their tensor product:

\[
V_\lambda \otimes V_\mu = \bigoplus \nu V_\nu^{\oplus c_{\lambda \mu}^\nu}
\]

\( c_{\lambda \mu}^\nu \) – **Littlewood-Richardson coefficients**, the number of isotypic components.
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**Theorem [Littlewood-Richardson, 1934]** The coefficient $c_\nu^\lambda \mu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$. 
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\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
1 & 3 & 3 \\
\end{array}
\]

(LR tableaux of shape \((7, 4, 3)/(3, 1)\) and type \((4, 3, 2)\). \( c_{(3,1)(4,3,2)}^{(7,4,3)} = 2 \))
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$S_n$ tensor products decomposition (diagonal action):

$$S_\lambda \otimes S_\mu = \bigoplus \nu\vdash_n S_\nu \bigoplus (\ldots)$$
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**$S_n$ tensor products decomposition** (diagonal action):

$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} S_\nu \oplus g(\lambda, \mu, \nu)$$

**Kronecker coefficients:**  $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$

E.g.: $S_{(2,1)} \otimes S_{(2,1)} = S_{(3)} \oplus S_{(2,1)} \oplus S_{(1,1,1)}$ and so $g((2, 1), (2, 1), \nu) = 1$ for $\nu = (3), (2, 1), (1, 1, 1)$. 

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$GL_N(\mathbb{C})$ acts on $V_\lambda, V_\mu$ and their tensor product:

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In terms of $GL(\mathbb{C}^m)$ modules $V_\lambda, V_\mu, V_\nu$ (Schur-Weyl duality):

$$\text{Sym}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \bigoplus \lambda, \mu, \nu g(\lambda, \mu, \nu) V_\lambda \otimes V_\mu \otimes V_\nu$$
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Plethysm coefficients in $GL$-representation compositions:

$$GL_N \to GL(V_\mu) \to GL(V_\nu) \iff GL_N \to V_\nu[V_\mu] = \bigoplus \lambda V_\lambda^{\oplus a_\lambda(\nu[\mu])}$$
The Algebraic Combinatorics problems

Problem (Murnaghan, 1938, then Stanley et al)

*Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $O_{\lambda, \mu, \nu}$ s.t. $g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}$. Alternatively, show that KRON ("Input: $(\lambda, \mu, \nu)$, output: $g(\lambda, \mu, \nu)$") is in $\#P$.*
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Classical motivation: (Littlewood–Richardson: for \( c^\nu_{\lambda, \mu} \), 
\( O_{\lambda, \mu, \nu} = \{ \text{LR tableaux of shape } \nu/\mu, \text{ type } \lambda \} \))

Theorem [Murnaghan] If \(|\lambda| + |\mu| = |\nu|\) and \( n > |\nu| \), then

\[
g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^\nu_{\lambda, \mu}.
\]
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Theorem [Murnaghan] If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^{\nu}_{\lambda, \mu}.$$ 

Modern motivation:
1. A positive combinatorial formula " $\iff$ " Computing Kronecker coefficients is in $\#P$.
2. Geometric Complexity Theory.
3. Invariant Theory, moment polytopes [see Bürgisser, Christandl, Mulmuley, Walter, Oliveira, Garg, Wigderson etc]
The Algebraic Combinatorics problems

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Results since then:
Combinatorial formulas for \( g(\lambda, \mu, \nu) \), when:

- \( \mu \) and \( \nu \) are hooks \( (\usebox{hooks}) \), [Remmel, 1989]
- \( \nu = (n - k, k) \) \( (\usebox{hooks}) \) and \( \lambda_1 \geq 2k - 1 \), [Ballantine–Orellana, 2006]
- \( \nu = (n - k, k) \), \( \lambda = (n - r, r) \) [Remmel–Whitehead, 1994; Blasiak–Mulmuley–Sohoni, 2013]
- \( \nu = (n - k, 1^k) \) \( (\usebox{hooks}) \), [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].
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Bounds and positivity:

[Pak-P]: \( g(\lambda, \mu, \mu) \geq |\chi^\lambda(2\mu_1 - 1, 2\mu_2 - 3, \ldots)| \) when \( \mu = \mu^T \). Corollaries:

\[
g(\lambda, \mu, \mu) > c \frac{2^{\sqrt{2k}}}{k^{9/4}} \quad \text{for} \quad \lambda = (|\mu| - k, k), \quad \text{and} \quad \text{diag}(\mu) \geq \sqrt{k}.
\]

[Saxl conjecture]: For every \( n > 9 \) there exists a self-conjugate partition \( \lambda \vdash n \), s.t. \( g(\lambda, \lambda, \mu) > 0 \) for all \( \mu \vdash n \). When \( n = \binom{m+1}{2} \), then \( \lambda = (m, m - 1, \ldots, 1) \). [Partial results: Pak-P-Vallejo, Ikenmeyer, Luo–Sellke]

Complexity results:

[Bürgisser-Ikenmeyer]: KRON is in GapP.

( Littlewood-Richardson, i.e. KRON’s special case, is \( \#P \)-complete )

[Pak-P]: If \( \nu \) is a hook, then KronPositivity is in P. If \( \lambda, \mu, \nu \) have fixed length there exists a linear time algorithm for deciding \( g(\lambda, \mu, \nu) > 0 \).

[Ikenmeyer-Mulmuley-Walter]: KronPositivity is NP-hard.

[Bürgisser-Christandl-Mulmuley-Walter]: membership in the moment polytope is NP and coNP.
Basic properties and formulas

From representation theory:

\[ g(\lambda, \lambda, (n)) = g(\lambda, \lambda', (1^n)) = 1 \]

Semigroup property: If \( \alpha, \beta, \gamma, \lambda, \mu, \nu \) are such that \( g(\alpha, \beta, \gamma) > 0 \) and \( g(\lambda, \mu, \nu) > 0 \) then \( g(\alpha + \lambda, \beta + \mu, \gamma + \nu) \geq \max\{g(\alpha, \beta, \gamma), g(\lambda, \mu, \nu)\} \)
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Via Schur functions \( s_\lambda \):

\[
s_\lambda(x) = \sum_{T: \text{SSYT}, \text{sh}(T) = \lambda} x^T
\]

\[
s_\lambda[ x \cdot y ] = \sum_{\mu, \nu} g(\lambda, \mu, \nu) s_\mu(x) s_\nu(y)
\]

x_1 y_1, x_1 y_2, ..., x_2 y_1, ...

Triple Cauchy identity:

\[
\prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z)
\]

A GAP formula via Contingency Arrays: (in [Christandl-Doran-Walter, Pak-Panova])

\[
g(\alpha, \beta, \gamma) = \sum_{\sigma^1, \sigma^2, \sigma^3 \in S_\ell} \text{sgn}(\sigma^1 \sigma^2 \sigma^3) CA(\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3),
\]

\[
CA(u, v, w) = \text{is # of } \ell \times \ell \times \ell \text{ contingency arrays } [A_{i,j,k}] \in \mathbb{N}^{k \times k \times k}:
\]

\[
\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_i, \quad \sum_{i,j} A_{i,j,k} = w_k
\]
“Example”: when $\nu = (n - k, k) – two rows

\[ \ell(\nu) = 2: \]

\[ g(\lambda, \mu, \nu) = \sum_{\sigma \in S_2} \text{sgn}(\sigma) \sum_{\alpha^i \vdash \nu_i - i + \sigma_i, i = 1, 2} c_{\alpha^1 \alpha^2}^\lambda c_{\alpha^1 \alpha^2}^\mu \]

\[ = \sum_{\alpha \vdash k, \beta \vdash n-k} c_{\alpha \beta}^\lambda c_{\alpha \beta}^\mu a_k(\lambda, \mu) \]

\[ - \sum_{\alpha \vdash k-1, \beta \vdash n-k+1} c_{\alpha \beta}^\lambda c_{\alpha \beta}^\mu a_{k-1}(\lambda, \mu) \]

**Corollary (Pak-P, Vallejo)**

The sequence $a_0(\lambda, \mu), a_1(\lambda, \mu), \ldots, a_n(\lambda, \mu)$ is unimodal for all $\lambda, \mu \vdash n$, i.e.

\[ a_0(\lambda, \mu) \leq a_1(\lambda, \mu) \leq \ldots \leq a_{\lfloor n/2 \rfloor}(\lambda, \mu) \geq \ldots \geq a_n(\lambda, \mu). \]
When $\nu = (n - k, k)$ – two rows

$$p_n(\ell, m) = \# \{ \lambda \vdash n, \ell(\lambda) \leq \ell, \lambda_1 \leq m \}$$

$$\sum_{n \geq 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m + \ell}{m}_q$$
When \( \nu = (n - k, k) \) – two rows

\[
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**Theorem (Sylvester 1878, Cayley’s conjecture 1856)**

The sequence \( p_0(\ell, m), \ldots, p_{\ell m}(\ell, m) \) is unimodal, i.e.

\[
p_0(\ell, m) \leq p_1(\ell, m) \leq \cdots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) \geq \cdots \geq p_{\ell m}(\ell, m)
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\[
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\]

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\]

“I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.”

J.J. Sylvester, 1878.
When $\nu = (n-k, k) - \text{two rows}$

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*The sequence $p_0(\ell, m), \ldots, p_\ell m(\ell, m)$ is unimodal, i.e.*

$$p_0(\ell, m) \leq p_1(\ell, m) \leq \ldots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) \geq \ldots \geq p_\ell m(\ell, m)$$

**Proof via Kronecker:**[Pak-P]

$$0 \leq g(\lambda, \mu, \nu) = \sum_{\alpha \vdash k, \beta \vdash n-k} c_{\alpha \beta}^\lambda c_{\alpha \beta}^\mu - \sum_{\alpha \vdash k-1, \beta \vdash n-k+1} c_{\alpha \beta}^\lambda c_{\alpha \beta}^\mu$$

$$a_k(\lambda, \mu) = \sum_{\alpha \vdash k, \beta \vdash m\ell-k} 1(\beta_i = m - \alpha_{\ell+1-i}, i = 1 \ldots \ell) = p_k(\ell, m)$$

+Corollary – $a_k(\lambda, \mu)$ unimodal

More corollaries: strict unimodality via semigroup property, exponential lower bounds via characters...
(Boolean) Complexity

**Input:** string of $n$ bits, i.e. $\text{size}(input) = n$.

**Decision problems:**

Is there an object, s.t.... ?

P = solution can be found in time $\text{Poly}(n)$

NP = solution can be verified in $\text{Poly}(n)$ (polynomial witness)

NP–Complete = in NP, and every NP problem can be reduced to it poly time;

**Counting problems:**

Compute $F(input) =$?

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#P = NP counting analogue; informally $F(input)$ counts Exp-many objects, whose verification is in P.
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- **P**: solution can be found in time \( \text{Poly}(n) \)
- **NP**: solution can be verified in \( \text{Poly}(n) \) (polynomial witness)
- **NP-Complete**: in NP, and every NP problem can be reduced to it poly time;

Counting problems:

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- **#P**: NP counting analogue; informally \( F(input) \) counts Exp-many objects, whose verification is in P.

The P vs NP Millennium Problem:

Is \( P = NP \)? Algebraic version: is \( VP = VNP \)?
(Boolean) Complexity

**Input:** string of $n$ bits, i.e. $\text{size}(\text{input}) = n$.

**Decision problems:**

Is there an object, s.t.... ?

$\mathbf{P}$ = solution can be found in time $\text{Poly}(n)$

$\mathbf{NP}$ = solution can be verified in $\text{Poly}(n)$ (polynomial witness)

$\mathbf{NP}$ –Complete = in $\mathbf{NP}$, and every $\mathbf{NP}$ problem can be reduced to it poly time;

**Counting problems:**

Compute $F(\text{input}) =$?

$\mathbf{FP}$ = solution can be found in time $\text{Poly}(n)$

$\mathbf{NP}$ = NP counting analogue; informally – $F(\text{input})$ counts Exp-many objects, whose verification is in $\mathbf{P}$.

The $\mathbf{P}$ vs $\mathbf{NP}$ Millennium Problem:

Is $\mathbf{P} = \mathbf{NP}$? Algebraic version: is $\mathbf{VP} = \mathbf{VNP}$?

An approach [Mulumley, Sohoni]: **Geometric Complexity Theory**
VP vs VNP: determinant vs permanent

**Arithmetic Circuits:**

\[
y = 3x_1 + x_1x_2
\]

Polynomials \( f_n \in \mathbb{F}[X_1, \ldots, X_n] \). Circuit – nodes are +, \( \times \) gates, input – \( X_1, \ldots, X_n \) and constants from \( \mathbb{F} \).

**Class VP** (Valliant’s P):
polynomials that can be computed with \( \text{poly}(n) \) large circuit (size of the associated graph).

**Class VNP**:
the class of polynomials \( f_n \), s.t. \( \exists g_n \in \text{VP} \) with
\[
f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).
\]
VP vs VNP: determinant vs permanent

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f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).
\]

**Theorem**[Bürgisser]: If VP = VNP, then P = NP if \( \mathbb{F} \) - finite or the Generalized Riemann Hypothesis holds.
VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial \( p(X) \) there exists some \( n \) s.t.

\[
p(X) = \det(A),
\]

where \( A = [\ell_{i,j}(X)]_{i,j=1}^n \) with \( \ell_{i,j}(X) \in \{a_0 + a_1 X_1 + \cdots + a_k X_k | a_i \in \mathbb{F}\} \).

The smallest \( n \) possible is the determinantal complexity \( dc(p) \).

Example: \( p = x_1^2 + x_1 x_2 + x_2 x_3 + 2x_1 \), then

\[
p = \det \begin{bmatrix} x_1 + 2 & x_2 \\ -x_3 + 2 & x_1 + x_2 \end{bmatrix}, \quad dc(p) = 2
\]
VP vs VNP: determinant vs permanent

**Universality of the determinant** [Cohn, Valiant]:
For every polynomial $p(X)$ there exists some $n$ s.t.

$$p(X) = \det(A),$$

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The smallest $n$ possible is the *determinantal complexity* $\text{dc}(p)$.

**Theorem:** [Valiant]  $p \in \text{VP} \iff \text{dc}(p) - \text{poly in deg}(p), k$. 


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The permanent:

$$\text{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^m X_{i,\sigma(i)}$$

**Theorem:** [Valiant] $\text{per}_m$ is VNP-complete.

**Conjecture** (Valiant, $\text{VP} \neq \text{VNP}$ equivalent)

$dc(\text{per}_m)$ *grows superpolynomially in* $m$. 
VP vs VNP: determinant vs permanent

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The smallest $n$ possible is the *determinantal complexity* $dc(p)$.

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**Theorem:** [Valiant] $\text{per}_m$ is VNP-complete.

**Conjecture (Valiant, VP $\neq$ VNP equivalent)**

$d\text{c}(\text{per}_m)$ grows superpolynomially in $m$.

**Known:** $d\text{c}(\text{per}_m) \leq 2^m - 1$ (Grenet 2011), $d\text{c}(\text{per}_m) \geq \frac{m^2}{2}$ (Mignon, Ressayre, 2004).

**Ryser’s formula:**

$$\text{per}_m(X) = (-1)^m \sum_{S \subseteq [1..m]} (-1)^{|S|} \prod_{i=1}^m (\sum_{j \in S} X_{i,j})$$
Geometric Complexity Theory

$GL_N$ action on polynomials:
$A \in GL_N(\mathbb{C}), \, \nu := (X_1, \ldots, X_N), \, f \in \mathbb{C}[X_1, \ldots, X_N],$
then $A f = f(A^{-1}\nu)$ (replaces variables with linear forms)

$GL_{n^2}\det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – determinant orbit.

$\Omega_n := \overline{GL_{n^2}\det_n}$ - determinant orbit closure.

$\text{per}_n^m := (X_1, 1)^{n-m}\text{per}_m$ – the padded permanent.
Geometric Complexity Theory

$GL_N$ action on polynomials:
$A \in GL_N(\mathbb{C}), \; v := (X_1, \ldots, X_N), \; f \in \mathbb{C}[X_1, \ldots, X_N],$
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$GL_{n^2}\det_n := \{ g \cdot \det_n \mid g \in GL_{n^2} \}$ – determinant orbit.

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Proposition (Lower bounds via geometry)
If $\text{per}_n^m \notin \overline{GL_{n^2}\det_n}$, then $dc(\text{per}_m) > n.$
**Geometric Complexity Theory**

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**Proposition ( Lower bounds via geometry )**

If \( \text{per}_n \not\in GL_{n^2}\det_n \), then \( \text{dc}(\text{per}_m) > n. \)

**Conjecture (GCT: Mulmuley and Sohoni)**

\[ \max\{ n : \text{per}_n \not\in GL_{n^2}\det_n \} (\leq \text{dc}(\text{per}_m)) \text{ grows superpolynomially.} \]

\[ \text{per}_m \in GL_{n^2}\det_n \iff GL_{n^2}\text{per}_m \subseteq GL_{n^2}\det_n. \]

\[ =:\Gamma^n_m \subseteq \Omega_n. \]
Geometric Complexity Theory

Proposition (Lower bounds via geometry)
If \( \text{per}_m^n \not\in \overline{GL}_n \text{det}_n \), then \( \text{dc}(\text{per}_m) > n \).

Conjecture (GCT: Mulmuley and Sohoni)
\( \max\{n : \text{per}_m^n \not\in \overline{GL}_n \text{det}_n\}(\leq \text{dc}(\text{per}_m)) \) grows superpolynomially.

\[ \text{per}_m^n \in \overline{GL}_n \text{det}_n \iff \overline{GL}_n \text{per}_m^n \subseteq \overline{GL}_n \text{det}_n \]
Proper Proposition (Lower bounds via geometry)
If \( \operatorname{per}^n_m \notin \text{GL}_{n^2} \text{det}_n \), then \( \text{dc}(\operatorname{per}^n_m) > n \).

Conjecture (GCT: Mulmuley and Sohoni)
\[
\max \{ n : \operatorname{per}^n_m \notin \text{GL}_{n^2} \text{det}_n \} \leq \text{dc}(\operatorname{per}^n_m) \quad \text{grows superpolynomially.}
\]

\[
\operatorname{per}^n_m \in \text{GL}_{n^2} \text{det}_n \iff \text{GL}_{n^2} \operatorname{per}^n_m \subseteq \text{GL}_{n^2} \text{det}_n
\]

Exploit the symmetry! Coordinate rings as \( \text{GL}_{n^2} \) representations:
\[
\mathbb{C}[\text{GL}_{n^2} \text{det}_n]_d \cong \bigoplus_{\lambda \vdash nd} V^{\delta_{\lambda, d, n}}, \quad \mathbb{C}[\text{GL}_{n^2} \operatorname{per}^n_m]_d \cong \bigoplus_{\lambda \vdash nd} V^{\gamma_{\lambda, d, n, m}}.
\]

Definition (Representation theoretic obstruction)
If \( \delta_{\lambda, d, n} < \gamma_{\lambda, d, n, m} \), then \( \lambda \) is a representation theoretic obstruction. Its existence shows \( \text{GL}_{n^2} \operatorname{per}^n_m \not\subseteq \text{GL}_{n^2} \text{det}_n \) and so \( \text{dc}(\operatorname{per}^n_m) > n \)!
(Non)existence of obstructions

\[ \mathbb{C}[GL_n^2 \det_n]_d \simeq \bigoplus_{\lambda \vdash nd} V^{\delta_{\lambda,d,n}}_\lambda, \]

\[ \mathbb{C}[GL_n^2 \text{per}_m]_d \simeq \bigoplus_{\lambda \vdash nd} V^{\gamma_{\lambda,d,n,m}}_\lambda, \]

Obstructions \( \lambda \): if \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \) for \( n > \text{poly}(m) \) \( \Rightarrow \) \( \mathsf{VP} \neq \mathsf{VNP} \).
(Non)existence of obstructions

\[\mathbb{C}[GL_n^2 \det_n]_d \cong \bigoplus_{\lambda \vdash nd} V^\lambda \oplus \delta_{\lambda,d,n}^{}, \quad \mathbb{C}[GL_n^2 \per_m^n]_d \cong \bigoplus_{\lambda \vdash nd} V^\lambda \oplus \gamma_{\lambda,d,n,m}^{},\]

**Obstructions** \(\lambda\): if \(\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}^{}\) for \(n > \text{poly}(m)\) \(\implies\) \(\text{VP} \neq \text{VNP}\).

**Conjecture (GCT: Mulmuley-Sohoni)**

*There exist representation theoretic obstructions that show superpolynomial lower bounds on \(\text{dc}(\text{per}_m)\).*
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2}\text{det}_n]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2}\text{per}_m]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

**Obstructions** $\lambda$: if $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ for $n > \text{poly}(m)$ $\implies$ $\text{VP} \neq \text{VNP}$.

**Conjecture (GCT: Mulmuley-Sohoni)**

*There exist representation theoretic obstructions that show superpolynomial lower bounds on* $\text{dc}(\text{per}_m)$.

If also $\delta_{\lambda,d,n} = 0$, then $\lambda$ is an **occurrence obstruction**.

**Conjecture (Mulmuley and Sohoni)**

*There exist occurrence obstructions that show superpolynomial lower bounds on* $\text{dc}(\text{per}_m)$.
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2 \det}]_d \cong \bigoplus_{\lambda \vdash nd} V^\lambda_{\delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2 \per}]_d \cong \bigoplus_{\lambda \vdash nd} V^\lambda_{\gamma_{\lambda,d,n,m}}, \]

Obstructions \( \lambda \): if \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \) for \( n > \text{poly}(m) \) \( \implies \) \( \text{VP} \neq \text{VNP} \).

Conjecture (GCT: Mulmuley-Sohoni)

There exist representation theoretic obstructions that show superpolynomial lower bounds on \( \text{dc(\per)} \).

If also \( \delta_{\lambda,d,n} = 0 \), then \( \lambda \) is an occurrence obstruction.

Conjecture (Mulmuley and Sohoni)

There exist occurrence obstructions that show superpolynomial lower bounds on \( \text{dc(\per)} \).

Theorem (Bürgisser-Ikenmeyer-P(FOCS’16, JAMS’18))

This Conjecture is false. There are no such occurrence obstructions.
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2}\det_n]_d \cong \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2}\per_m]_d \cong \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}}, \]

**Obstructions** \( \lambda \): if \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \) for \( n > \text{poly}(m) \) \( \implies \) \( \text{VP} \neq \text{VNP} \).

What are these \( \delta_{\lambda,d,n} \) and \( \gamma_{\lambda,d,n,m} \)?

**Kronecker coefficients:**

\[ \delta_{\lambda,d,n} \leq sk(\lambda, n^d) \leq g(\lambda, n^d, n^d) \]

(Symmetric Kronecker: \( sk(\lambda, \mu) := \dim \text{Hom}_{S_{|\lambda|}}(S^\lambda, S^2(S^\mu)) = \text{mult}_\lambda \mathbb{C}[GL_{n^2}\det_n]_d \))

**Plethysm coefficients:** of \( GL \).

\[ a_\lambda(d[n]) := \text{mult}_\lambda Sym^d(Sym^n(V)) \geq \gamma_{\lambda,d,n,m}. \]
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2 \text{det}}]_d \cong \bigoplus_{\lambda \vdash nd} V_\lambda^{\delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2 \text{per}}]_d \cong \bigoplus_{\lambda \vdash nd} V_\lambda^{\gamma_{\lambda,d,n,m}}, \]

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**Conjecture (GCT, Mulmuley and Sohoni)**

*There exist \( \lambda \), s.t. \( g(\lambda, n^d, n^d) = 0 \) and \( \gamma_{\lambda,d,n,m} > 0 \) for some \( n > \text{poly}(m) \).*
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2\det_n}]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2\per_m}]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

**Obstructions** \( \lambda \): if \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \) for \( n > \text{poly}(m) \) \( \implies \) \( \text{VP} \neq \text{VNP} \).

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**Plethysm coefficients:** of \( GL \).

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**Conjecture (GCT, Mulmuley and Sohoni)**

There exist \( \lambda \), s.t. \( g(\lambda, n^d, n^d) = 0 \) and \( \gamma_{\lambda,d,n,m} > 0 \) for some \( n > \text{poly}(m) \).

**Theorem (Ikenmeyer-P (FOCS’16, Adv.Math.’17))**

Let \( n > 3m^4, \lambda \vdash nd \). If \( g(\lambda, n^d, n^d) = 0 \) (so \( mult_\lambda \mathbb{C}[GL_{n^2\det_n}] = 0 \)), then \( mult_\lambda(\mathbb{C}[GL_{n^2(X_{1,1})^{n-m}\per_m}]) = 0 \).


For any \( \rho \), let \( n \geq |\rho|, d \geq 2, \lambda := (nd - |\rho|, \rho) \). Then \( g(\lambda, n \times d \times n \times d) \geq a_\lambda(d[n]) \).
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2 \cdot \text{det} n}]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2 \cdot \text{per} m}]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

Obstructions \( \lambda \): if \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \) for \( n > \text{poly}(m) \) \( \implies \) \( \text{VP} \neq \text{VNP} \).

Conjecture (Mulmuley and Sohoni 2001)

For all \( c \in \mathbb{N}_{\geq 1} \), for infinitely many \( m \), there exists a partition \( \lambda \) occurring in \( \mathbb{C}[GL_{n^2 X_{11}^{n-m} \cdot \text{per} m}] \) but not in \( \mathbb{C}[GL_{n^2 \cdot \text{det} n}] \), where \( n = m^c \).

Theorem (B"urgisser-Ikenmeyer-P (FOCS’16, JAMS’18))

Let \( n, d, m \) be positive integers with \( n \geq m^{25} \) and \( \lambda \vdash nd \). If \( \lambda \) occurs in \( \mathbb{C}[GL_{n^2 X_{11}^{n-m} \cdot \text{per} m}] \), then \( \lambda \) also occurs in \( \mathbb{C}[GL_{n^2 \cdot \text{det} n}] \). In particular, the Conjecture is false, there are no “occurrence obstructions”.
No occurrence obstructions I: positive Kroneckers

Theorem (Ikenmeyer-P)

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n \times d, n \times d) = 0$, then

$$\text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1}^{n-m}\text{per}_m)]) = 0.$$  

Proof:

$\bar{\lambda} := (\lambda_2, \lambda_3, \ldots) \vdash |\lambda| - \lambda_1$

Theorem (Kadish-Landsberg)

If $\text{mult}_\lambda \mathbb{C}[GL_{n^2}X_{11}^{n-m}\text{per}_m] > 0$, then $|\bar{\lambda}| \leq md$ and $\ell(\lambda) \leq m^2$.

Theorem (Degree lower bound, [IP] )

If $|\bar{\lambda}| \leq md$ with $a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.  

Greta Panova
No occurrence obstructions I: positive Kroneckers

**Theorem (Ikenmeyer-P)**

Let \( n > 3m^4 \), \( \lambda \vdash nd \). If \( g(\lambda, n \times d, n \times d) = 0 \), then
\[
\text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(X_{1,1})^{n-m}\text{per}_m]) = 0.
\]

**Proof:**
\[
\bar{\lambda} := (\lambda_2, \lambda_3, \ldots) \vdash |\lambda| - \lambda_1
\]

**Theorem (Kadish-Landsberg)**

If \( \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}X_{11}^{n-m}\text{per}_m]) > 0 \), then \( |\bar{\lambda}| \leq md \) and \( \ell(\lambda) \leq m^2 \).

**Theorem (Degree lower bound, [IP] )**

If \( |\bar{\lambda}| \leq md \) with \( a_\lambda(d[n]) > g(\lambda, n \times d, n \times d) \), then \( d > \frac{n}{m} \).

**Theorem (Kronecker positivity, [IP] )**

Let \( \lambda \vdash dn \). Let \( \mathcal{X} := \{(1), (2 \times 1), (4 \times 1), (6 \times 1), (2, 1), (3, 1)\} \).

(a) If \( \bar{\lambda} \in \mathcal{X} \), then \( a_\lambda(d[n]) = 0 \).

(b) If \( \bar{\lambda} \notin \mathcal{X} \) and \( m \geq 3 \) such that \( \ell(\lambda) \leq m^2 \), \( |\bar{\lambda}| \leq md \), \( d > 3m^3 \), and \( n > 3m^4 \), then
\[
g(\lambda, n \times d, n \times d) > 0.
\]
Kronecker positivity I: hook-like $\lambda$s

Proposition (Ikenmeyer-P)

If there is an $a$, such that $g(\nu^k(a^2), a \times a, a \times a) > 0$ for all $k$, s.t. $k \notin H^1(\rho)$ and $a^2 - k \notin H^2(\rho)$ for some sets $H^1(\rho), H^2(\rho) \subset [\ell, 2a + 1]$, then $g(\nu^k(b^2), b \times b, b \times b) > 0$ for all $k$, s.t. $k \notin H^1(\rho)$ and $b^2 - k \notin H^2(\rho)$ for all $b \geq a$.

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{k : g(\nu^k(c^2), c \times c, c \times c) > 0\}$, we have
Claim: Suppose that $k \in P_c$, then $k, k + 2c + 1 \in P_{c+1}$. 
Kronecker positivity I: hook-like $\lambda$s

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If there is an $a$, such that $g(\nu^k(a^2), a \times a, a \times a) > 0$ for all $k$, s.t. $k \not\in H^1(\rho)$ and $a^2 - k \not\in H^2(\rho)$ for some sets $H^1(\rho), H^2(\rho) \subset [\ell, 2a + 1]$, then $g(\nu^k(b^2), b \times b, b \times b) > 0$ for all $k$, s.t. $k \not\in H^1(\rho)$ and $b^2 - k \not\in H^2(\rho)$ for all $b \geq a$.

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{ k : g(\nu^k(c^2), c \times c, c \times c) > 0 \}$, we have
Claim: Suppose that $k \in P_c$, then $k, k + 2c + 1 \in P_{c+1}$.

Corollary
We have that $g(\lambda, h \times w, h \times w) > 0$ for $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for most “small” partitions $\rho$ and all but finitely many values of $j$. 

Greta Panova
Kronecker positivity II: squares, and decompositions

Theorem (Ikenmeyer-P)

Let $\nu \notin \mathcal{X}$ and $\ell = \max(\ell(\nu) + 1, 9)$, $a > 3\ell^{3/2}$, $b \geq 3\ell^2$ and $|\nu| \leq ab/6$. Then $g(\nu(ab), a \times b, a \times b) > 0$.

Proof sketch: decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k - 1) \times k) + \sum_{k=2}^{\ell} y_k((k - 1) \times 2).$$
Kronecker positivity II: squares, and decompositions

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Let $\nu \notin \mathcal{X}$ and $\ell = \max(\ell(\nu) + 1, 9)$, $a > 3\ell^{3/2}$, $b \geq 3\ell^2$ and $|\nu| \leq ab/6$. Then $g(\nu(ab), a \times b, a \times b) > 0$.

**Proof sketch:** decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k - 1) \times k) + \sum_{k=2}^{\ell} y_k((k - 1) \times 2).$$

**Crucial facts:**

- $g(k \times k, k \times k, k \times k) > 0$ [Bessenrodt-Behns].
- Transpositions: $g(\alpha, \beta, \gamma) = g(\alpha, \beta^T, \gamma^T)$ (with $\beta = \gamma = w\times h$)
- Hooks and exceptional cases: $g(\lambda, h \times w, h \times w) > 0$ for all $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for $|\rho| \leq 6$ and almost all $j$s.
- Semigroup property for positive triples:
  $$g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max(g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)).$$
Kronecker vs plethysm: inequality of multiplicities

**Stability** [Manivel]: \( g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d) \), as \( n \to \infty \).

\( \text{St}^1(\rho) := \{(n, d) \mid g((nd - |\rho|, \rho), n \times d, n \times d)\} = a_\rho(d) \).
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**Proposition (Ikenmeyer-P)**

*Fix \( \rho, \) and let \( (n, d) \in \text{St}^1(\rho), \) which is true in particular if \( n \geq |\rho|. \) Let \( \lambda = (nd - |\rho|, \rho). \) Then \( g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n]). \)
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**Proof:** \( \lambda = \mu + d(n - m) \). Suppose \( g(\lambda, n \times d, n \times d) < a_\lambda(d[n]) \):

**KL’14:** If \( \mu \vdash md \) then \( mult_{\mu + d(n-m)}(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq a_\mu(d[m]) \), where \( V_m := Sym^m\mathbb{C}^{m^2} \).
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Kronecker vs plethysm: inequality of multiplicities

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**Stability:** \( g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d). \)

**GCT:** If \( \text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq g(\lambda, n \times d, n \times d) \) then \( dc(f_m) > n \) for some \( f_{m,n} \in V_m. \)
Kronecker vs plethysm: inequality of multiplicities

Stability [Manivel]: $g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d)$, as $n \to \infty$.
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Proposition (Ikenmeyer-P)

Fix $\rho$, and let $(n, d) \in St^1(\rho)$, which is true in particular if $n \geq |\rho|$. Let $\lambda = (nd - |\rho|, \rho)$. Then $g(\lambda, n \times d, n \times d) \geq a_\lambda(\lambda)$.

Proof: $\lambda = \mu + d(n - m)$. Suppose $g(\lambda, n \times d, n \times d) < a_\lambda(\mu)$:

KL'14: If $\mu \vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[GL_n^2(X_1,1)^{n-m}V_m]) \geq a_\mu(d[m])$, where $V_m := \text{Sym}^m \mathbb{C}^{m^2}$.

Stability: $g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d)$.

GCT: If $mult_{\lambda}(\mathbb{C}[GL_n^2(X_1,1)^{n-m}V_m]) \geq g(\lambda, n \times d, n \times d)$ then $dc(f_m) > n$ for some $f_{m,n} \in V_m$.

$\implies mult_{\lambda}(\mathbb{C}[GL_n^2(X_1,1)^{n-m}V_m]) \geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$

$\implies \max_{f \in V_m} dc(f_{m,n}) > n \to \infty$
Thank you!

Algebraic Geometry

\[ [X_{1,1}X_{2,2} - X_{1,2}X_{2,1}] \]

Representation Theory

Statistical Mechanics/

Complexity Theory

P vs NP

Algebraic Combinatorics

\[ s_{(2,2)}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 \]

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1 & 1 & 2 \\
2 & 3 & 3 \\
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2 & 2 & 2 \\
3 & 3 & 3 \\
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2 & 3 & 3 \\
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