On the identifiability of symmetric tensors beyond the Kruskal's bound

Elena Angelini

Dip. di Ingegneria dell'Informazione e Scienze Matematiche
Università degli Studi di Siena

Joint works with L. Chiantini, A. Mazzon, N. Vannieuwenhoven

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PRELIMINARIES

- Basics on symmetric tensors
- Kruskal's criterion for symmetric tensors
- The Hilbert function for finite sets in $\mathbb{P}^n$
- The Cayley-Bacharach property for finite sets in $\mathbb{P}^n$

BEYOND KRUSKAL'S CRITERION FOR FORMS

- Symmetric tensors in $S^d\mathbb{C}^3$
- The case of ternary optics
- Symmetric tensors in $S^d\mathbb{C}^{n+1}$

REFERENCES

Notation

- d, $n \in \mathbb{N}$
- $\mathbb{C}^{n+1}$: {linear forms in $x_0, \dots, x_n$ / $\mathbb{C}$}
- $S^d\mathbb{C}^{n+1}$: {forms of degree d in $x_0, \dots, x_n$ / $\mathbb{C}$}
- $T \in S^d\mathbb{C}^{n+1} \rightsquigarrow [T] \in \mathbb{P}(S^d\mathbb{C}^{n+1}) \cong \mathbb{P}^N$, $N = \binom{n+d}{d} - 1$
- $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ *Veronese embedding* of $\mathbb{P}^n$ of degree d
  $\nu_d([a_0x_0 + \ldots + a_nx_n]) = [(a_0x_0 + \ldots + a_nx_n)^d]$
- $A = \{P_1, \dots, P_{\ell(A)}\} \subset \mathbb{P}^n \rightsquigarrow \nu_d(A) = \{\nu_d(P_1), \dots, \nu_d(P_{\ell(A)})\}$
  $\langle \nu_d(A) \rangle$: linear space spanned by $\nu_d(P_1), \dots, \nu_d(P_{\ell(A)})$

Definition

A finite set $A \subset \mathbb{P}^n$ *computes* $T \in \mathbb{P}^N$ if $T \in \langle \nu_d(A) \rangle$.

Definition

A finite set A which computes T is *non-redundant* if, for any $A' \subsetneq A$, $T \notin \langle \nu_d(A') \rangle$.

Remark

A non-redundant $\Longrightarrow \dim\langle \nu_d(A) \rangle = \ell(A) - 1$

Definition

The *rank* of T is $r = \min\{\ell(A) \mid T \in \langle \nu_d(A) \rangle\}$.

Definition

A finite set $A \subset \mathbb{P}^n$ *computes the rank* of T if A computes T, it is non-redundant and $\ell(A) = r$.

Definition

T of rank r is *identifiable* if there exists a unique A computing the rank of T.

Definition

The *d-th Kruskal's rank* of a finite set $A \subset \mathbb{P}^n$ is

$$k_d(A) = \max\{k \mid \forall A' \subset A,\ \ell(A') \leq k,\ \dim\langle \nu_d(A')\rangle = \ell(\nu_d(A')) - 1\}$$

Remark

- $k_d(A) \leq \min\{N+1, \ell(A)\}$
- $k_d(A)$ maximal $\Longrightarrow k_d(A')$ maximal, $\forall A' \subset A$
- A general $\Longrightarrow k_d(A) = \min\{N+1, \ell(A)\}$

Theorem (Kruskal's criterion, 2017 [COV2])

Let $T \in \mathbb{P}^N$, $d \geq 3$, let $A \subset \mathbb{P}^n$ be a non-redundant set computing T. Assume that $d = d_1 + d_2 + d_3$, with $d_1 \geq d_2 \geq d_3 \geq 1$. If

$$\ell(A) \leq \frac{k_{d_1}(A) + k_{d_2}(A) + k_{d_3}(A) - 2}{2} \tag{1}$$

then T has rank $\ell(A)$ and it is identifiable.

Definition

The *evaluation map* of degree d on $Y = \{Y_1, \ldots, Y_\ell\} \subset \mathbb{C}^{n+1}$ is

$$ev_Y(d) : S^d\mathbb{C}^{n+1} \longrightarrow \mathbb{C}^\ell$$

$$ev_Y(d)(F) = (F(Y_1), \ldots, F(Y_\ell)).$$

Definition

Let Y be a set of homogeneous coordinates for $Z \subset \mathbb{P}^n$ finite set. The *Hilbert function* of Z is the map

$$h_Z : \mathbb{Z} \longrightarrow \mathbb{N}$$

$$h_Z(j) = 0, \text{ for } j < 0, \; h_Z(j) = rank(ev_Y(j)), \text{ for } j \geq 0.$$

Definition

The *first difference of the Hilbert function* Dh_Z of Z is

$$Dh_Z(j) = h_Z(j) - h_Z(j-1), \; j \in \mathbb{Z}.$$

Elementary properties of h_Z and Dh_Z

- $Dh_Z(j) = 0$, for $j < 0$
- $h_Z(0) = Dh_Z(0) = 1$
- $Dh_Z(j) \geq 0$, for all j
- $h_Z(i) = \sum_{0 \leq j \leq i} Dh_Z(j)$
- $h_Z(j) = \ell(Z)$, for all $j \gg 0$
- $Dh_Z(j) = 0$, for $j \gg 0$
- $\sum_j Dh_Z(j) = \ell(Z)$

Proposition

- $Z' \subset Z \Longrightarrow h_{Z'}(j) \leq h_Z(j)$, $Dh_{Z'}(j) \leq Dh_Z(j)$
- $Dh_Z(i) \leq i$, $\exists\, i > 0 \Longrightarrow Dh_Z(i) \geq Dh_Z(i+1)$

Proposition (A., Chiantini, Vannieuwenhoven 2018, [ACV])

Let $T \in S^d\mathbb{C}^{n+1}$ and let $A, B \subset \mathbb{P}^n$ be non-redundant finite sets computing T. Then $Dh_{A\cup B}(d+1) > 0$.

Proposition (A., Chiantini, Vannieuwenhoven 2018, [ACV])

Let $A, B \subset \mathbb{P}^n$ be finite sets. Then, for any $d \in \mathbb{N}$,

$$\dim(\langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle) = -1 + \ell(A\cap B) + \sum_{j\geq d+1} Dh_{A\cup B}(j). \quad (2)$$

Theorem (Davis 1985, [Da])

Let $Z \subset \mathbb{P}^2$ be a finite set s.t.

- $Dh_Z(j) = j + 1$ for $j \in \{0, \dots, i-1\}$ and $Dh_Z(i) \leq i$;
- $Dh_Z(j_0) = Dh_Z(j_0 + 1) = e$, $\exists\, j_0 \geq i - 1$.

Then $Z = Z_1 \cup Z_2$ where Z_1 lies on a curve of degree e of $\mathbb{P}^2$ and

$$Dh_{Z_2}(j) = Dh_Z(e + j) - e,\ j \in \{0, \dots, j_0 - e - 1\}.$$

Proposition (A., Chiantini 2019, [AC])

Let $T \in S^d\mathbb{C}^3$ with a non-redund. dec. $A \subset \mathbb{P}^2$. Then, there is no other $B \subset \mathbb{P}^2$ non-redund. dec. of T with $A \cap B = \emptyset$, $\ell(B) \leq \ell(A)$ and s.t.:

- $Dh_{A\cup B}(j) = j + 1$ $j \in \{0, \dots, i-1\}$ and $Dh_{A\cup B}(i) \leq i$;
- $Dh_{A\cup B}(j_0) = Dh_{A\cup B}(j_0 + 1) = e < i$, $\exists\, j_0 > i - 1$.

Definition

A finite set $Z \subset \mathbb{P}^n$ satisfies the *Cayley-Bacharach property in degree d*, $CB(d)$, if, for all $P \in Z$, it holds that every form of degree d vanishing at $Z \setminus \{P\}$ also vanishes at P.

Examples

- $Z \subset \mathbb{P}^2$ set of 6 general points
  $\Longrightarrow Dh_Z(j) = j + 1,\ 0 \leq j \leq 2$, Z has $CB(1)$ but not $CB(2)$
- $Z \subset \mathbb{P}^2$ set of 6 points on an irreducible conic
  $\Longrightarrow Dh_Z(0) = 1,\ Dh_Z(1) = Dh_Z(2) = 2,\ Dh_Z(3) = 1$
  Z has $CB(2)$ and $CB(1)$
- $Z \subset \mathbb{P}^2$ set of 6 points, 5 aligned
  $\Longrightarrow Dh_Z(0) = 1, Dh_Z(1) = 2, Dh_Z(j) = 1, 2 \leq j \leq 4$
  Z has not $CB(1)$

Theorem (A., Chiantini, Vannieuwenhoven 2018, [ACV])

If a finite set $Z \subset \mathbb{P}^n$ has $CB(d)$, then, for any $j \in \{0, \ldots, d+1\}$,

$$Dh_Z(0) + \ldots + Dh_Z(j) \leq Dh_Z(d+1-j) + \ldots + Dh_Z(d+1). \quad (3)$$

Proposition (A., Chiantini, Mazzon 2018, [ACM])

Let $T \in S^d\mathbb{C}^{n+1}$ and let $A, B \subset \mathbb{P}^n$ be non-redundant finite sets computing T, with $A \cap B = \emptyset$. Then $Z = A \cup B$ satisfies $CB(d)$.

Proof.

- ✓ $\exists\ P \in Z \mid h_Z(d) = \sum_{i=0}^{d} Dh_Z(i) > h_{Z\setminus\{P\}}(d) = \sum_{i=0}^{d} Dh_{Z\setminus\{P\}}(i)$
- ✓ $\ell(Z) = \sum_{i=0}^{\infty} Dh_Z(i) = 1 + \ell(Z \setminus \{P\}) = \sum_{i=0}^{\infty} Dh_{Z\setminus\{P\}}(i) \Longrightarrow \sum_{i\geq d+1} Dh_Z(i) = \sum_{i\geq d+1} Dh_{Z\setminus\{P\}}(i)$
- ✓ (2): $\dim(\langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle) = \dim(\langle\nu_d(A \setminus \{P\})\rangle \cap \langle\nu_d(B \setminus \{P\})\rangle)$
- ✓ either A or B is redundant for T $\rightarrow | \leftarrow$

Theorem (A., Chiantini 2019, [AC])

Let $T \in \mathbb{P}(S^d\mathbb{C}^3)$ and let $A \subset \mathbb{P}^2$ be a non-redundant finite set computing T. The form T is identifiable of rank $\ell(A)$ if one of the following occurs:

- $d = 2m$
  - $k_{m-1}(A) = \min\{\binom{m+1}{2}, \ell(A)\}$
  - $h_A(m) = \ell(A) \leq \binom{m+2}{2} - 2$
- $d = 4m+1$
  - $k_{2m}(A) = \min\{\binom{2m+2}{2}, \ell(A)\}$
  - $h_A(2m+1) = \ell(A) \leq \binom{2m+2}{2} + m$
- $d = 4m+3$
  - $k_{2m+1}(A) = \min\{\binom{2m+3}{2}, \ell(A)\}$
  - $h_A(2m+2) = \ell(A) \leq \binom{2m+3}{2} + m$

Sketch of the proof for the case $d = 2m$ (I).

By induction on $r = \ell(A) > 1$.
Let $B = \{P'_1, \dots, P'_{\ell(B)}\} \subset \mathbb{P}^2$ be another non-red. dec. of T s.t. $\ell(B) \leq \ell(A)$ and set $Z = A \cup B$:

- ✓ $A \cap B \neq \emptyset$, $s = \ell(A \cap B) > 0$,

$$T = \sum_{j=1}^{r} a_j \nu_d(P_j) = \sum_{j=1}^{s} b_j \nu_d(P_j) + \sum_{j=s+1}^{\ell(B)} b_j \nu_d(P'_j)$$

$$T_0 = \sum_{j=1}^{s} (a_j - b_j) \nu_d(P_j) + \sum_{j=s+1}^{r} a_j \nu_d(P_j) = \sum_{j=s+1}^{\ell(B)} b_j \nu_d(P'_j)$$

T_0 is computed by A, $B_0 = B \setminus A$ with $A \cap B_0 = \emptyset$

- ▷ B_0 is non-red. for T_0, otherwise $\rightarrow \mid \leftarrow$

Sketch of the proof for the case $d = 2m$ (II).

▷ A red. for $T_0 \implies A' = \{P_{q+1}, \ldots, P_r\} \subset A$, with $q \leq s$, non-red. dec. of T_0 s.t. $\ell(A') = r' < r$, $h_{A'}(m) = \ell(A')$, $k_{m-1}(A') = \min\{\binom{m+1}{2}, r'\}$ $\rightarrow | \leftarrow$

✓ $A \cap B = \emptyset \implies Z \;\; CB(d)$

- $r \leq \binom{m+1}{2}$
  $\ell(Z) \leq 2r = 2k_{m-1}(A) = 2\sum_{i=0}^{m-1} Dh_A(i) \leq 2\sum_{i=0}^{m-1} Dh_Z(i) \leq \sum_{i=0}^{m-1} Dh_Z(i) + \sum_{i=m+2}^{d+1} Dh_Z(i) \leq \ell(Z) - Dh_Z(m) - Dh_Z(m+1)$
  $\rightarrow | \leftarrow$
- $r > \binom{m+1}{2}$
  $Dh_Z(m) + Dh_Z(m+1) \leq \ell(Z) - 2\sum_{i=0}^{m-1} Dh_A(i) \leq 2r - h_A(m-1) = 2r - \binom{m+1}{2} < 2m$
  $\rightarrow | \leftarrow$

$\square$

Theorem (A., Chiantini 2019, [AC])

Let $T \in \mathbb{P}(S^d\mathbb{C}^3)$ and let $A \subset \mathbb{P}^2$ be a non-redundant finite set computing T. The decomposition A computes the rank of T if one of the following occurs:

- $d = 2m$
  - $h_A(m) = \ell(A)\,(\leq \binom{m+2}{2})$
- $d = 4m+1$
  - $k_{2m}(A) = \min\{\binom{2m+2}{2}, \ell(A)\}$
  - $h_A(2m+1) = \ell(A) \leq \binom{2m+2}{2} + m$
- $d = 4m+3$
  - $k_{2m+1}(A) = \min\{\binom{2m+3}{2}, \ell(A)\}$
  - $h_A(2m+2) = \ell(A) \leq \binom{2m+3}{2} + m + 1$

Assume that $d = 8$ ($m = 4$) and $n = 2$. Consider:

- ⋆ $T \in \mathbb{P}(S^8\mathbb{C}^3) \cong \mathbb{P}^{44}$
- ⋆ $A = \{P_1, \dots, P_{\ell(A)}\} \subset \mathbb{P}^2$ dec. of T
  - ✓ A is non-redundant
  - ✓ $k_3(A) = \min\{10, \ell(A)\}$
  - ✓ $h_4(A) = \ell(A)$

Remarks

- ▷ $\ell(A) \leq 13 \Longrightarrow T$ is identifiable of rank $\ell(A)$
- ▷ $\ell(A) \in \{14, 15\} \Longrightarrow T$ has rank $\ell(A)$
  - ✓ $\ell(A) = 15 \Longrightarrow$ the number of dec. of the general T is 16, [RS]

$\ell(A) = 14$?

- ⋆ $A = \{P_1, \ldots, P_{14}\} \subset \mathbb{P}^2$ dec. of T
  - ✓ A is non-redundant
  - ✓ $k_3(A) = 10$
  - ✓ $h_4(A) = 14$
- ⋆ $A^\vee = \{P_1^\vee, \ldots, P_{14}^\vee\} \subset (\mathbb{P}^2)^\vee$ dual set of A

Remarks

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j	0	1	2	3	4	5	...
$h_A(j)$	1	3	6	10	14	14	...
$Dh_A(j)$	1	2	3	4	4	0	...

- $0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^4 \longrightarrow J_{A^\vee} \longrightarrow 0$

$$M = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \\ L_{13} & L_{14} & L_{15} & L_{16} \end{pmatrix}$$

- $\star$ $B = \{P'_1, \ldots, P'_{\ell(B)}\} \subset \mathbb{P}^2$ another dec. of T
  - ✓ B is non-redundant
  - ✓ $\ell(B) = 14$
- $\star$ $Z = A \cup B$

Remarks

- Z has $CB(9)$, otherwise $A \cap B \neq \emptyset$ and B doesn't exist $\rightarrow | \leftarrow$
-

j	0	1	2	3	4	5	6	7	8	9	10	...
$Dh_Z(j)$	1	2	3	4	4	4	4	3	2	1	0	...

- $\ell(Z) = 28$, $A \cap B = \emptyset$, $Z = (4, 7)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^{\oplus 4} \arrow[r] & J_{A^\vee} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-11) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-7) \arrow[r] \arrow[u] & J_{Z^\vee} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 4} \xrightarrow{SM} \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)^{\oplus 4} \to J_{B^\vee} \to 0$$

$$SM = \begin{pmatrix} q'_1 & q'_2 & q'_3 & q'_4 \\ L_1 & L_5 & L_9 & L_{13} \\ L_2 & L_6 & L_{10} & L_{14} \\ L_3 & L_7 & L_{11} & L_{15} \\ L_4 & L_8 & L_{12} & L_{16} \end{pmatrix} \rightsquigarrow q'_1 = q'_3 = 0, \;\; q'_2 \neq 0 \; q'_4 \neq 0$$

$$B^\vee \leftrightsquigarrow S \in \mathbb{P}(H^0(J_{A^\vee}(7))/(S^3(\mathbb{C}^3) \otimes H^0(J_{A^\vee}(4)))) \cong \mathbb{P}^{11}$$

- $\langle \nu_8(A)\rangle \cong \mathbb{P}^{13} \subset \mathbb{P}^{44}$
- $\mathbb{P}(H^0(J_{A^\vee}(8))), \mathbb{P}(H^0(J_{B^\vee}(8))) \cong \mathbb{P}^{30} \subset (\mathbb{P}^{44})^\vee$
- $\mathbb{P}(H^0(J_{A^\vee \cup B^\vee}(8))) \cong \mathbb{P}^{17} \subset (\mathbb{P}^{44})^\vee$

Theorem (A., Chiantini 2019, [AC])

The map $f : \mathbb{P}^{11} \dashrightarrow \langle \nu_8(A)\rangle$ defined by

$$f(S) = \mathbb{P}(H^0(J_{A^\vee}(8)) + H^0(J_{B(S)^\vee}(8)))^\vee.$$

is birational.

Corollary

- ⋆ $T \in S^8\mathbb{C}^3$ general unidentifiable of rank 14 $\Rightarrow$ T has 2 dec. computing its rank
- ⋆ $\langle \nu_8(A)\rangle$ contains a $\mathbb{P}^{11}$ of tensors with 2 dec. computing the rank

The algorithm

INPUT: $A = \{P_1, \ldots, P_{14}\} \subset \mathbb{P}^2 \leftrightsquigarrow A^\vee = \{P_i^\vee = [v_i]\}_{i=1}^{14} \subset (\mathbb{P}^2)^\vee$
$T = \sum_{i=1}^{14} \lambda_i \nu_8(P_i) = [(p_0, \ldots, p_{44})]^\vee$

PROCEDURE: check that

1) $\dim\langle \nu_8(v_1), \ldots, \nu_8(v_{14})\rangle = 14$ ✓
2) $h_4(A) = 14$ ✓
3) $k_3(A) = 10$ ✓
4) the 13×12 matrix of the linear system $(p_0, \ldots, p_{44}) \cdot A_2 = 0_{1\times 13}$ has rank 12 ✓

OUTPUT: T has rank 14 and is identifiable

Theorem (A., Chiantini 2019, [AC])

Let $T \in \mathbb{P}(S^d\mathbb{C}^{n+1})$ and let $A \subset \mathbb{P}^n$ be a non-redundant finite set computing T, with $d \geq 3$ and $h_A(1) = \min\{n+1, r\}$. The form T is identifiable of rank $\ell(A)$ if one of the following occurs:

- $d = 2m$
  - $h_A(m-1) \geq \ell(A) - \min\left\{\frac{n-1}{2}, \frac{m-1}{2}\right\}$
- $d = 2m+1$
  - $k_m(A) = \ell(A)$
  - $h_A(m-1) \geq \ell(A) - \min\left\{\frac{n-1}{2}, \frac{m-1}{2}\right\}$

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