Max-Planck-Institut für
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Sum of Squares: exercises

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1. Recall that a set $S \subset \mathbb{R}^n$ is *convex* if

$$x, y \in S \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in S \quad \forall \lambda \in [0,1],$$

and a function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \forall x, y \in \mathbb{R}^n, \quad \lambda \in [0,1].$$

(a) Are the following sets convex? Justify as needed.

$$\begin{aligned} S_1 &:= \{(x,y) \in \mathbb{R}^2 : x + y \geq 1\} \\ S_2 &:= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\} \\ S_3 &:= S_1 \cap S_2, \\ S_4 &:= S_1 \cup S_2 \end{aligned}$$

What can you say in general about unions and intersections of convex sets?

(b) Are the following functions convex? Justify as needed.

$$\begin{aligned} f_1(x) &= 1/(1+x^2), \\ f_2(x,y) &= x^2 \cdot (1 + 2y - x^2), \\ f_3(x,y) &= \sin(x^2+y^2) - \cos(x+y). \end{aligned}$$

(c) Show that a function is convex if and only if its *epigraph*

$$\operatorname{epi} f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$$

is a convex set.

2. Consider the linear programming (LP) problem:

$$\text{minimize } x_1 - x_2 \qquad \text{s.t.} \begin{cases} x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{cases}$$

(a) Write this problem as an LP in *standard form*. What are the corresponding matrices A, b, c?

(b) Write down the corresponding dual problem.

(c) What are the optimal solutions of the primal and dual problems?

(d) Verify that strong duality holds for this example.

(e) Implement this in your favorite LP solver.

3. Consider the following semidefinite programming problem:

$$\text{minimize } x + 2y \qquad \text{subject to } \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0.$$

(a) Sketch the feasible set. Is it convex?

(b) Write the dual semidefinite program.

(c) What are the optimal solutions?

(d) What can you say about strong duality?

4. Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex body (i.e., full-dimensional, compact) that includes the origin in its interior. Its *polar* convex body is defined as

$$\mathcal{C}^\circ = \{y \in \mathbb{R}^n : \langle y, x \rangle \leq 1 \quad \forall x \in \mathcal{C}\}.$$

   (a) Show that $\mathcal{C}^\circ$ is a convex body (i.e., convex, compact, full-dimensional).

   (b) Let $\mathcal{C}$ be a triangle with vertices $(-1, 1)$, $(-1, -1)$ and $(a, 0)$, where $a > 0$. Draw $\mathcal{C}$ and $\mathcal{C}^\circ$, as a function of the parameter a.

   (c) Let $\mathcal{C}$ be an axis-aligned ellipse of semiaxes a and b. What is $\mathcal{C}^\circ$?

   (d) Let $\mathcal{C} = \{x \in \mathbb{R}^n : \|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1\}$, where $p \geq 1$. Find a "nice" description of $\mathcal{C}^\circ$. Hint: use Hölder's inequality.

5. In this exercise we explore the polar set from a computational point of view.

   (a) Assume that $\mathcal{C}$ is the feasible set of an SDP, i.e.,

$$\mathcal{C} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \preceq C\},$$

   where A_i and C are given symmetric matrices, and $C \succ 0$. Find a convenient description of $\mathcal{C}^o$ in terms of semidefinite programming. Can you optimize a linear function over $\mathcal{C}^o$?

   (b) Compute the polar of the 3×3 elliptope, i.e., the set

$$\mathcal{C} = \{X \in \mathcal{S}^3_+ : X_{11} = X_{22} = X_{33} = 1\}$$

   Plot $\mathcal{C}$ and $\mathcal{C}^\circ$.

6. Using semidefinite programming, give a description of the convex hull of two disjoint disks in $\mathbb{R}^3$. In particular, consider

$$D_1 = \{(x, y, z) \in \mathbb{R}^3 \, : \, (x-2)^2 + y^2 \leq 1, \quad z = 0\}$$

and

$$D_2 = \{(x, y, z) \in \mathbb{R}^3 \, : \, (x+1)^2 + z^2 \leq 1, \quad y = 0\}.$$

Implement your construction using an SDP solver. Plot D_1 and D_2, and the resulting convex hull.

7. Consider the polynomial $p(x) = x^4 + 2ax^2 + b$. For what values of (a, b) is this polynomial nonnegative? Draw the region of nonnegativity in the (a, b) plane. Where does the discriminant of p vanish? How do you explain this?

8. Let $M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ be the Motzkin polynomial. Show that $M(x, y, z)$ is not SOS, but $(x^2 + y^2 + z^2) \cdot M(x, y, z)$ is.

9. Give a rational certificate of the nonnegativity of the trigonometric polynomial $p(\theta) = 5 - \sin\theta + \sin 2\theta - 3\cos 3\theta$.

10. Consider a univariate polynomial of degree d, that is bounded by one in absolute value on the interval $[-1, 1]$. How large can its leading coefficient be? Give an SOS formulation for this problem, solve it numerically for $d = 2, 3, 4, 5$, and plot the found solutions. Can you guess what the general solution is as a function of d? Can you characterize the optimal polynomial?

11. Consider a given univariate rational function $r(x)$, for which we want to find a good polynomial approximation $p(x)$ of fixed degree d on the interval $[-2, 2]$.

   (a) Write an SOS formulation to compute the best polynomial approximation of $r(x)$ in the supremum norm.

   (b) Same as before, but now $p(x)$ is also required to be convex.

   (c) Same as before, but $p(x)$ is required to be a convex lower bound of $r(x)$ (i.e., $p(x) \leq r(x)$ for all $x \in [-2, 2]$).

   (d) Let $r(x) = \frac{1-2x+x^2}{1+x+x^2}$. Find the solution of the previous subproblems (for $d = 4$), and plot them.

12. In general, the SOS decomposition of a univariate polynomial is not unique. Given a specific basis of $\mathbb{R}[x]$ (for instance, the standard monomial basis we have been considering), a "natural" choice can be obtained by finding a matrix Q satisfying

$$p(x) = [x]_d^T \, Q \, [x]_d, \qquad Q \succeq 0$$

and such that the determinant of Q is as large as possible. This "central solution" can be computed by solving a convex optimization problem, since $\log\det Q$ is a concave function on the region where Q is positive semidefinite. In fact, this convex optimization problem can be reformulated a semidefinite programming problem.

(a) Compute numerically the central solution for the polynomial $p(x) = x^6 - 6x^5 + 16x^4 - 24x^3 + 22x^2 - 12x + 4$.

(b) Show, using the KKT optimality conditions, that in general the inverse of the optimal matrix Q is a Hankel matrix.

13. Consider the polynomial system $\{x + y^3 = 2, x^2 + y^2 = 1\}$.

(a) Is it feasible over $\mathbb{C}$? How many solutions are there?

(b) Is it feasible over $\mathbb{R}$? If not, give a Positivstellensatz-based infeasibility certificate of this fact.

14. Consider the butterfly curve in $\mathbb{R}^2$, defined by the equation

$$x^6 + y^6 = x^2.$$

Give an sos certificate that the real locus of this curve is contained in a disk of radius 5/4. Is this the best possible constant?

15. Consider the quartic form in four variables

$$p(w, x, y, z) := w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz.$$

(a) Show that $p(w, x, y, z)$ is not a sum of squares.

(b) Find a multiplier $q(w, x, y, z)$ such that $q(w, x, y, z)p(w, x, y, z)$ is a sum of squares.

16. Consider a random variable X, with an unknown probability distribution supported on the set Ω, and for which we know its first $d+1$ moments $(\mu_0, \ldots, \mu_d)$. We want to find bounds on the probability of an event $S \subseteq \Omega$, i.e., want to bound $P(X \in S)$. We assume S and Ω are given intervals. Consider the following optimization problem in the decision variables c_k:

$$\min \sum_{k=0}^{d} c_k \mu_k \qquad \text{s.t.} \qquad \begin{cases} \sum_{k=0}^{d} c_k x^k \geq 1 & \forall x \in S \\ \sum_{k=0}^{d} c_k x^k \geq 0 & \forall x \in \Omega. \end{cases} \tag{1}$$

(a) Show that any feasible solution of (1) gives a valid upper bound on $P(X \in S)$. How would you solve this problem?

(b) Assume that $\Omega = [0, 5]$, $S = [4, 5]$, and we know that the mean and variance of X are equal to 1 and 1/2, respectively. Give upper and lower bounds on $P(X \in S)$. Are these bounds tight? Can you find the worst-case distributions?

17. The *stability number* $\alpha(G)$ of a graph G is the cardinality of its largest stable set. Define the ideal $I = \langle x_i(1-x_i) \quad i \in V, \quad x_i x_j \quad (i,j) \in E \rangle$.

(a) Show that $\alpha(G)$ is *exactly* given by

$$\min \gamma \qquad \gamma - \sum_{i \in V} x_i \quad \text{is SOS mod } I.$$

[Hint: recall (or prove!) that if I is zero-dimensional and radical, then $p(x) \geq 0$ on $V(I)$ if and only if $p(x)$ is SOS mod I.]

(b) Recall that a polynomial is 1-SOS if it can be written as a sum of squares of affine (degree 1) polynomials. Show that an upper bound on $\alpha(G)$ can be obtained by solving

$$\min \gamma \qquad \gamma - \sum_{i \in V} x_i \quad \text{is 1-SOS mod } I. \tag{2}$$

(c) Show that the given generators of the ideal I are already a Gröbner basis. Show that there is a natural bijection between standard monomials and stable sets of G.

(d) As a consequence of the previous fact, show that $\alpha(G)$ is equal to the degree of the Hilbert function of $\mathbb{R}[x]/I$.

Now let $G = (V, E)$ be the Petersen graph, given in Figure 1.

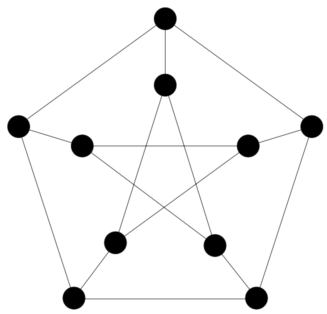

Figure 1: Petersen graph

(a) Find a stable set in the Petersen graph of maximum cardinality.

(b) Solve problem (2) for the Petersen graph. What is the corresponding upper bound?

(c) Compute the Hilbert function of I using Macaulay2, and verify that this answer is consistent with your previous results.

18. Consider linear maps between symmetric matrices, i.e., of the form $\Lambda : \mathcal{S}^n \to \mathcal{S}^m$. A map is said to be a *positive map* if it maps the PSD cone S_+^n into the PSD cone S_+^m (i.e., it preserves positive semidefinite matrices).

   (a) Show that any linear map of the form $A \mapsto \sum_i P_i^T A P_i$, where $P_i \in \mathbb{R}^{n \times m}$, is positive. These maps are known as *decomposable* maps.

   (b) Show that the linear map $C : \mathcal{S}^3 \to \mathcal{S}^3$ (due to M.-D. Choi) given by:

$$C : A \mapsto \begin{bmatrix} 2a_{11} + a_{22} & 0 & 0 \\ 0 & 2a_{22} + a_{33} & 0 \\ 0 & 0 & 2a_{33} + a_{11} \end{bmatrix} - A.$$

   is a positive map, but is not decomposable.

   Hint: Consider the polynomial defined by $p(x, y) := y^T \Lambda(xx^T) y$. How can you express positivity and decomposability of the linear map Λ in terms of the polynomial p?