NUMERICAL HODGE THEORY

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1. Introduction

These are rough[1] notes for my talk in *Summer School on Numerical Computing in Algebraic Geometry* on 14.08.18. For more details and references please see the paper *Computing periods of hypersurfaces* (arXiv:1803.08068). Soon there will be an upcoming paper with Pierre Lairez to continue the story to finding curves in surfaces (and algebraic cycles in hypersurfaces).

The general theme of this conference is to take a problem in algebraic geometry of impossible theoretical or computational complexity and to recast it into a million mindless computations in analysis, letting the computer deal with it. We apply this to a classical problem in geometry; that of computing algebraic cycles in a variety.

We stick to the "toy" problem of detecting curves in a quartic surface in $\mathbb{P}^3_{\mathbb{C}}$, which is nevertheless open. The tools are not limited to this case but can handle hypersurfaces of any dimension and degree, provided the computations terminate.

Curves are not interesting in this regard, since there are no algebraic cycles besides points and the curve itself. Surfaces of degrees one, two and three in $\mathbb{P}^3_{\mathbb{C}}$ are also not interesting because there is no change in behavior as the defining coefficients vary. In fact, algebraic cycles in surfaces of degree less than four have been completely understood a hundred years ago. The first non-trivial case is the classification of curves in a given quartic surface.

2. Curves in a quartic surface

Take a smooth quartic surface $X = Z(f) \subset \mathbb{P}^3_{\mathbb{C}}$ cut out by a homogeneous equation of degree four:

$$f \in \mathbb{C}[x, y, z, w]_4.$$

Problem. What kind of algebraic curves are there in the surface X?

For instance:

- How many lines are there in X? How do they intersect? (Easy to do symbolically. Give it a try!)
- How many quadric curves are there in X? (Symbolically hard. Maybe Bertini can do it?)
- How many twisted cubics are there in X? (Should be impossible to do it naively.)

There are always finitely many of these rational curves. However, if we were to ask for smooth planar cubics contained in X, we would realize that they come in families: If $C \subset X$ is a curve of degree three contained in a plane H then $H \cap X = C \cup L$ for a line L, by degree reasons. Then any other plane H' containing L will intersect X along L and another plane cubic C'.

The observation regarding plane cubics in X suggest that we should really consider two curves equivalent if one can be deformed into the other in some way. Otherwise, we may end up with continuously many curves in X. The notion of equivalence we have in mind is called "linear equivalence", this *roughly* declares two curves to be equivalent if they appear as the fibers of a

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[1] For example, instead of including references we point to your favorite search engine.

map $X \dashrightarrow \mathbb{P}^1$. (To see that the plane cubics C and C' above are linearly equivalent, project X away from the line L.)

Definition 2.1. Let $\mathrm{Pic}(X)$ be the formal $\mathbb{Z}$-linear combination of curves in X modulo linear equivalence.

It turns out that for any smooth surface in $\mathbb{P}^3$ there is a number $\rho(X)$ such that $\mathrm{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$. This number $\rho(X)$ is called the *Picard rank of* X.

Problem. Compute the Picard rank of X. Even better, compute how generators of $\mathrm{Pic}(X)$ intersect, to be represented by a $\rho(X) \times \rho(X)$ integral matrix.

Remark 2.2. Armed with the intersection matrix, one can compute all sorts of delicate information about a linear equivalence class. For instance, we can find smooth rational curves of specified degree by a combinatorial procedure.

Remark 2.3. This "toy" problem of determining $\rho(X)$ is open! There is no reasonable method to compute the Picard group for a given surface, even for a quartic surface.

Bringing analysis to this problem rests on an observation of Hodge (and Lefschetz and Picard). He joined topology and analysis to attack this algebraic problem.

3. Topology and analysis

An algebraic curve C in the algebraic surface X defines a homology class:

$$[C] \in \mathrm{H}_2(X, \mathbb{Z}).$$

Note that elements in the latter are topological 2-cycles modulo homological equivalence.

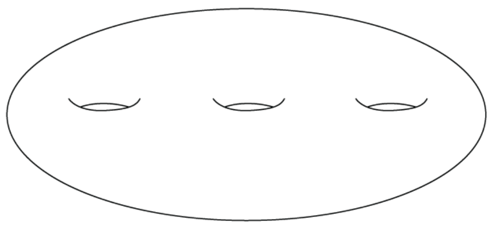

Figure 1. Underlying topological space of an algebraic curve (of genus 3).

Lemma 3.1. *For any smooth surface $X \subset \mathbb{P}^3$, the association above induces an* inclusion*:*

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{H}_2(X, \mathbb{Z}).$$

Remark 3.2. For *any* smooth quartic surface X in $\mathbb{P}^3_{\mathbb{C}}$ we have $\mathrm{H}_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$.

Consider the affine chart $\mathbb{C}^3 \subset \mathbb{P}^3_{\mathbb{C}}$ given by $w = 1$ and with coordinates x, y, z. Let ∂_z denote differentiation by z. Define the following two form on $\mathbb{C}^3$:

$$\omega_X = \frac{\mathrm{d}x\mathrm{d}y}{\partial_z f(x, y, z, 1)}. \tag{3.0.1}$$

Remark 3.3. The restriction of ω_X to X uniquely extends to a holomorphic two form on the entirety of X. Observing that $\mathrm{d}f = 0$ on X, one can show that (up to sign) the roles of x, y, z (and even of w) can be interchanged to obtain the same restriction of ω_X onto X.

Theorem 3.4 (Lefschetz (1,1))**.** *The Picard group of X is precisely the kernel of the following map:*

$$\mathrm{H}_2(X,\mathbb{Z}) \to \mathbb{C} : \gamma \mapsto \int_\gamma \omega_X. \tag{3.0.2}$$

Sketch. Suppose a class $\gamma \in \mathrm{H}_2(X,\mathbb{Z})$ supports an algebraic curve C. The (complex) tangent space of C is one dimensional at any point. Therefore, the two form $\mathrm{d}x\mathrm{d}y$ is annihilated the moment you restrict it to C. The integral $\int_\gamma \omega_X$ must vanish.

The argument above can be generalized to any dimension, however generalizing the opposite direction to any dimension would get you a million dollars (see Hodge conjecture). For codimension one algebraic cycles, as in curves in surfaces, the proof uses the long exact sequence associated to the exponential sequence. □

3.1. **Strategy.** The observation above suggests the following strategy to compute the Picard group of X.

(1) Compute explicit 2-cycles $\gamma_1, \ldots, \gamma_{22}$ generating $\mathrm{H}_2(X,\mathbb{Z})$.
(2) Compute, numerically or otherwise, the integral $\int_{\gamma_i} \omega_X$. The value of these integrals are called *periods* of X.
(3) Compute the kernel of the map (3.0.2).

Supposing that we got to the last step, we are faced with the problem that it is essentially impossible to *certifiably* find integral relations between transcendental numbers. Nevertheless, we will use Lenstra–Lenstra–Lovàsz (LLL) lattice reduction algorithm to get a good candidate for the kernel.

The second step does not pose a problem at all, although it might be time consuming to compute integrals numerically to high precision.

The first step is where we are derailed. It is difficult to compute such an explicit basis in practice for any given f defining X, let alone to automate this procedure.

4. Deforming the integrals

Take $Y = Z(g) \subset \mathbb{P}^3_{\mathbb{C}}$ be another smooth quartic defined by a simpler polynomial, say,

$$g = x^4 + y^4 + z^4 + w^4.$$

Consider the interpolation $h_t := (1-t)g + tf$ for $t \in \mathbb{C}$. This defines a family of quartic surfaces $X_t := Z(h_t) \subset \mathbb{P}^3_{\mathbb{C}}$, which are smooth except possibly for finitely many *singular values* of t, let us denote this finite set of bad values by $S \subset \mathbb{C}$.

Let $\omega(t)$ denote the following 2-form on the affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$ defined by $w = 1$, which is defined exactly as in (3.0.1):

$$\omega(t) := \frac{\mathrm{d}x\mathrm{d}y}{\partial_z h_t(x,y,z,1)}.$$

In light of Lefschetz (1,1) theorem, the kernels of the integration maps:

$$\mathrm{H}_2(X_t,\mathbb{Z}) \to \mathbb{C} : \gamma \mapsto \int_\gamma \omega(t),$$

are the Picard groups $\mathrm{Pic}(X_t)$.

Theorem 4.1 (Ehresmann fibration theorem)**.** *Locally for $t \in \mathbb{C} \setminus S$ the topological spaces underlying X_t do not change and can be identified. In particular, the groups $\mathrm{H}_2(X_t,\mathbb{Z})$ can be identified and this identification is unique locally in t.*

What this means for us is that if we start with a basis $\gamma_1^Y, \ldots, \gamma_{22}^Y \in \mathrm{H}_2(Y,\mathbb{Z}) = \mathrm{H}_2(X_0,\mathbb{Z})$ then there exists a (locally) unique family of bases $\gamma_1(t), \ldots, \gamma_{22}(t) \in \mathrm{H}_2(X_t,\mathbb{Z})$ such that $\gamma_i(0) = \gamma_i^Y$.

Definition 4.2. We will refer to the following function as the *period function of the family* X_t:

$$\mathfrak{p}(t) := \left(\int_{\gamma_1(t)} \omega(t), \dots, \int_{\gamma_{22}(t)} \omega(t) \right).$$

Remark 4.3. We are not bothered by the fact that $\gamma_i(t)$ is uniquely defined only locally since the following computations are local in t. The end result, i.e., the Picard rank does not care about which basis we may have ended up with on X_1.

4.1. **New strategy.** Details of how to actually carry out the following strategy is a little involved and we refer to the introduction of the *Computing periods of hypersurfaces* paper cited at the beginning.

- Compute a differential operator $\mathcal{D} \in \mathbb{C}(t)[\partial_t]$ such that $\mathcal{D} \cdot \mathfrak{p}(t) \equiv 0$.
- Since Y is simple, compute the initial conditions $\mathfrak{p}(0), \mathfrak{p}'(0), \mathfrak{p}''(0), \dots$.
- Solve the initial value problem specified by these two computations to obtain $\mathfrak{p}(1)$.

Try *period-suite* which implements precisely this procedure, it is available at `https://github.com/period-suite/period-suite`. Now that we have the periods of X we can find the kernel of the map (3.0.2) using LLL, although we sacrifice certainty in our answers. The command `HodgeLattice` will do this for you. See the `README` file of `period-suite` for further explanations.

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