Sinkhorn and power method for tensors with positive entries

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Overview

$$x, y, z \in \mathbb{R}^n_{++}, \quad A, B \in \mathbb{R}^{n\times n}_{++}, \quad R, S, T \in \mathbb{R}^{n\times n\times n}_{++}$$

1) Eigenvectors and singular vectors of positives matrices

$$Ax = \lambda x \qquad \begin{cases} Ay = \lambda x \\ Bx = \sigma y \end{cases}$$

2) Eigenvectors and singular vectors of positive third order tensors

$$Txx = \lambda x \qquad \begin{cases} Txy = \lambda x \\ Sxx = \sigma y \end{cases} \qquad \begin{cases} Tyz = \lambda x \\ Sxz = \sigma y \\ Rxy = \theta z \end{cases}$$

3) Generalizations

Nonlinear Overview

$$x, y, z \in \mathbb{R}^n_{++}, \quad A, B \in \mathbb{R}^{n\times n}_{++}, \quad R, S, T \in \mathbb{R}^{n\times n\times n}_{++}, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

1) Eigenvectors and singular vectors of positives matrices

$$Ax^\alpha = \lambda x \qquad \begin{cases} Ay^\beta = \lambda x \\ Bx^\alpha = \sigma y \end{cases}$$

2) Eigenvectors and singular vectors of positive third order tensors

$$Tx^\alpha x^\alpha = \lambda x \qquad \begin{cases} Tx^\alpha y^\beta = \lambda x \\ Sx^\alpha x^\alpha = \sigma y \end{cases} \qquad \begin{cases} Ty^\beta z^\gamma = \lambda x \\ Sx^\alpha z^\gamma = \sigma y \\ Rx^\alpha y^\beta = \theta z \end{cases}$$

3) Generalizations

$$Ax^{\alpha} = \lambda x$$

Hilbert projective metric on $\mathbb{R}^n_{++}$

Hilbert metric

$$\mu(x, y) = \max_{i,j} \; \log\left(\frac{x_i}{y_i}\frac{y_j}{x_j}\right) \qquad \forall x, y \in \mathbb{R}^n_{++}$$

- μ is a projective metric

$$\mu(s\,x,\; t\,y) = \mu(x, y) \qquad \forall s, t > 0$$

- Spectral problem $\iff$ Fixed point problem

$$Ax = \lambda x \qquad \iff \qquad \mu(Ax, x) = 0$$

Completeness of Hilbert metric

$$\mathbb{S}^n_{++} = \{x \in \mathbb{R}^n_{++} \mid \|x\| = 1\}$$

Theorem

$(\mathbb{S}_{++}, \mu)$ is a **complete metric space**

$\implies$ Banach fixed point theorem

Proof sketch:

$$U \subset \mathbb{R}^n_{++} \text{ closed} \quad \implies \quad \frac{1}{c}\|x-y\| \le \mu(x,y) \le c\|x-y\|$$

$$\forall x, y \in U \cap \mathbb{S}^n_{++}$$

Birkhoff-Hopf theorem

Birkhoff-Hopf theorem

Let $A \in \mathbb{R}^{m\times n}_{++}$, then

$$\mu(Ax, Ay) \leq \kappa(A)\, \mu(x, y) \qquad\qquad \forall x, y \in \mathbb{R}^n_{++}$$

where

$$\kappa(A) = \tanh\big(\Delta(A)/4\big) < 1 \qquad \text{and} \qquad \Delta(A) = \max_{i,j} \mu(Ae_i, Ae_j)$$

$$\Delta(A) = \Delta(A^\top) = \max_{i,j,k,l} \ln\Big(\frac{A_{ik}A_{jl}}{A_{il}A_{jk}}\Big)$$

Perron-Frobenius theorem

$A \in \mathbb{R}^{n\times n}_{++}$, $x_0 \in \mathbb{R}^{n}_{++}$

$$Ax = \lambda x \qquad\qquad x \in \mathbb{S}^n_+ \tag{$\star$}$$

Perron-Frobenius theorem

($\star$) **has a unique solution** $x^\star$ and $\mu(x_{k+1}, x^\star) \leq c\,\kappa(A)^k$ with
$c = d(x_1, x_0)(1-\kappa(A))^{-1}$

$$x_{k+1} = \frac{Ax_k}{\|Ax_k\|}$$

α-eigenvalues

$$Ax^{\alpha} = \lambda x$$

Lipschitz constant of $x \mapsto Ax^\alpha$

$A \in \mathbb{R}_{++}^{n\times n}$, $\alpha \in \mathbb{R}$

Dilatations

$$\alpha \in \mathbb{R} \qquad \Longrightarrow \qquad \mu(x^\alpha, y^\alpha) = |\alpha|\,\mu(x,y) \qquad \forall x, y > 0$$

$$\mu(Ax^\alpha, Ay^\alpha) \quad \leq \quad \kappa(A)\,\mu(x^\alpha, y^\alpha) \quad = \quad |\alpha|\kappa(A)\,\mu(x,y)$$

Perron-Frobenius theorem for α-eigenvalues

$A \in \mathbb{R}^{n\times n}_{++}$, $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}^n_{++}$

$$Ax^\alpha = \lambda x \qquad x \in \mathbb{S}^n_{++} \tag{$\star$}$$

Theorem

$$r = |\alpha|\kappa(A) < 1$$

$\implies$ $(\star)$ **has a unique solution** $x^\star$ and $\mu(x_k, x^\star) \leq c\, r^k$

$$x_{k+1} = \frac{Ax_k^\alpha}{\|Ax_k^\alpha\|}$$

$$\begin{cases} Ay^{\beta} = \lambda x \\ Bx^{\alpha} = \sigma y \end{cases}$$

Singular values of positive matrices

$A \in \mathbb{R}^{n\times n}_{++}$

$$\begin{cases} Ay &= \lambda\, x \\ A^\top x &= \lambda\, y \end{cases} \qquad (x,y) \in \mathbb{R}^{n+n}_{++}$$

$$\kappa\begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} = \sup_{(x,y),(\tilde{x},\tilde{y})\in\mathbb{R}^{n+n}_{++}} \frac{\mu\big[\,(Ay, A^\top x),\,(A\tilde{y}, A^\top\tilde{x})\,\big]}{\mu\big[\,(x,y),\,(\tilde{x},\tilde{y})\,\big]} = 1$$

$$\underbrace{\sup_{x,\tilde{x}\in\mathbb{R}^n_{++}} \frac{\mu(\,Ax,\ A\tilde{x}\,)}{\mu(\,x,\ \tilde{x}\,)}}_{=\kappa(A)} < 1 \qquad \underbrace{\sup_{y,\tilde{y}\in\mathbb{R}^n_{++}} \frac{\mu(\,A^\top y,\ A^\top\tilde{y}\,)}{\mu(\,y,\ \tilde{y})}}_{=\kappa(A^\top)=\kappa(A)} < 1$$

Consider **product metric** on $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ and **Lipschitz matrix**

Lipschitz matrix of singular value problem

$A \in \mathbb{R}^{m\times n}_{++}$, $\|\cdot\|$ a monotonic norm on $\mathbb{R}^2$

$$\begin{cases} Ay &= \lambda x \\ A^\top x &= \sigma y \end{cases} \qquad (x,y) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$$

$$\begin{bmatrix} \mu(Ay, A\tilde{y}) \\ \mu(A^\top x, A^\top \tilde{x}) \end{bmatrix} \le \underbrace{\begin{pmatrix} 0 & \kappa(A) \\ \kappa(A^\top) & 0 \end{pmatrix}}_{=:\ L} \begin{bmatrix} \mu(x, \tilde{x}) \\ \mu(y, \tilde{y}) \end{bmatrix}$$

$$\left\| \begin{bmatrix} \mu(Ay, A\tilde{y}) \\ \mu(A^\top x, A^\top \tilde{x}) \end{bmatrix} \right\| \le \underbrace{\left(\max_{v \neq 0} \frac{\|Lv\|}{\|v\|} \right)}_{\ge\ \rho(L)} \left\| \begin{bmatrix} \mu(x, \tilde{x}) \\ \mu(y, \tilde{y}) \end{bmatrix} \right\|$$

Lipschitz matrix and metrization of $\mathbb{R}^m_{++} \times \mathbb{R}^n_{++}$

$$\begin{bmatrix} \mu(Ay, A\tilde{y}) \\ \mu(A^\top x, A^\top \tilde{x}) \end{bmatrix} \leq L \begin{bmatrix} \mu(x, \tilde{x}) \\ \mu(y, \tilde{y}) \end{bmatrix} \qquad \text{with} \qquad L = \begin{pmatrix} 0 & \kappa(A) \\ \kappa(A) & 0 \end{pmatrix}$$

Lemma [G., Hein, Tudisco (2018)]

$$\mu_w\Big(\begin{pmatrix} Ay \\ A^\top x \end{pmatrix}, \begin{pmatrix} A\tilde{y} \\ A^\top \tilde{x} \end{pmatrix} \Big) \leq \rho(L)\, \mu_w\Big(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Big)$$

$$\mu_w\Big(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Big) = w_1\, \mu(x, \tilde{x}) + w_2\, \mu(y, \tilde{y})$$

where

$$w = (w_1, w_2) \in \mathbb{R}^2_{++} \qquad\qquad w\, L \; = \; \rho(L)\, w$$

Perron-Frobenius for singular values

$A \in \mathbb{R}^{n\times n}_{++}$, $(x_0, y_0) > 0$

$$\begin{cases} Ay &= \lambda x \\ A^\top x &= \sigma y \end{cases} \qquad (x,y) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++} \qquad (\star)$$

Theorem

As

$$\rho \begin{pmatrix} 0 & \kappa(A) \\ \kappa(A^\top) & 0 \end{pmatrix} = \kappa(A) < 1,$$

there **exists** a **unique** solution $(x^\star, y^\star)$ to $(\star)$ and

$$\mu(x_{k+1}, x^\star) + \mu(y_{k+1}, y^\star) \leq c\, \kappa(A)^k$$

$$(x_{k+1}, y_{k+1}) = \left(\frac{Ay_k}{\|Ay_k\|}, \frac{A^\top x_k}{\|A^\top x_k\|}\right)$$

Nonlinear singular values of positive matrices

$$\begin{cases} Ay^{\beta} & = & \lambda x \\ Bx^{\alpha} & = & \sigma y \end{cases} \qquad x, y \in \mathbb{S}^{n}_{++}$$

Nonlinear singular values of positive matrices

$A, B \in \mathbb{R}_{++}^{n\times n}$

$$\begin{cases} Ay^{\beta} &= \lambda x \\ Bx^{\alpha} &= \sigma y \end{cases} \qquad x, y \in \mathbb{S}_{++}^{n}$$

- **Matrix operator norm:** $M \in \mathbb{R}_{++}^{n\times n}, p, q \in (1, \infty)$

$$\|M\|_{p,q} = \max_{x\neq 0} \frac{\|Mx\|_{q'}}{\|x\|_p} = \max_{x,y\neq 0} \frac{\langle Mx, u\rangle}{\|x\|_p \|u\|_q}$$

with $\alpha = \frac{1}{p-1}, \beta = \frac{1}{q-1}$.

  * NP-hard for general $M \in \mathbb{R}^{n\times n}$ and $(p-1)(q-1) < 1$

[Steinberg (2005)]

- **Sinkhorn equation** with $\alpha = \beta = -1$

Lipschitz Matrix of nonlinear singular values

$$\begin{bmatrix} \mu(Ay^{\beta}, A\tilde{y}^{\beta}) \\ \mu(Bx^{\alpha}, B\tilde{x}^{\alpha}) \end{bmatrix} \leq \underbrace{\begin{pmatrix} 0 & \kappa(A) \\ \kappa(B) & 0 \end{pmatrix} \begin{pmatrix} |\alpha| & 0 \\ 0 & |\beta| \end{pmatrix}}_{:=L} \begin{bmatrix} \mu(x, \tilde{x}) \\ \mu(y, \tilde{y}) \end{bmatrix}$$

$$\mu_w\Big(\begin{pmatrix} Ay^{\beta} \\ A^{\top} x^{\alpha} \end{pmatrix}, \begin{pmatrix} A\tilde{y}^{\beta} \\ A^{\top} \tilde{x}^{\alpha} \end{pmatrix} \Big) \leq \rho(L)\, \mu_w\Big(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Big)$$

$$\rho(L) = \underbrace{\sqrt{\kappa(A)\kappa(B)|\alpha\beta|}}_{\text{contraction rate}} \qquad w = \underbrace{(\sqrt{\kappa(B)|\alpha|}, \sqrt{\kappa(A)|\beta|})}_{\text{metric weights}}$$

Perron-Frobenius for matrix (α, β)-singular values

$A, B \in \mathbb{R}^{n\times n}_{++}$, $\alpha, \beta \in \mathbb{R}$, $(x_0, y_0) > 0$

$$\begin{cases} Ay^{\alpha} & = & \lambda x \\ Bx^{\beta} & = & \sigma y \end{cases} \qquad x, y \in \mathbb{S}^n_{++} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$r = \sqrt{\kappa(A)\kappa(B)|\alpha\beta|} < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution** $(x^\star, y^\star)$ and
$\mu(x_{k+1}, x^\star) + \mu(y_{k+1}, y^\star) \leq c\, r^k$

$$(x_{k+1}, y_{k+1}) = \Big(\frac{Ay_k^{\alpha}}{\|Ay_k^{\alpha}\|}, \frac{Bx_k^{\beta}}{\|Bx_k^{\beta}\|}\Big)$$

$\ell^{p,q}$-norm of positive matrices

Corollary [G., Hein, Tudisco (2018)]

If $A \in \mathbb{R}^{n\times n}_{++}$ and $\kappa(A)^2 < (p-1)(q-1)$, then $\|A\|_{p,q}$ can be computed by power method with linear convergence rate.

- Condition is tight for

$$\|A_\varepsilon\|_{p,q} = \max_{x\neq 0} \frac{\|A_\varepsilon x\|_{q'}}{\|x\|_p} \qquad \text{with} \qquad p, q \in (1,\infty),\ A_\varepsilon = \begin{pmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{pmatrix}, \varepsilon > 0$$

For $\varepsilon = 3/4$,

Classical condition ([1], [2], [3]) $\Longrightarrow$ $1 \leq (p-1)(q-1)$

$\kappa(A_\varepsilon)^2 < (p-1)(q-1)$ $\Longrightarrow$ $0.04 < (p-1)(q-1)$

1: Boyd, (1973), 2: Bhaskara, Vijayaraghavan, (2011), 3: G., Hein, (2016)

- Application to Markov chains (convergence of MCMC algorithms)

Positive tensors of order 3

Notation

$$T \in \mathbb{R}^{n \times n \times n} \qquad x, y \in \mathbb{R}^n \qquad Txy \in \mathbb{R}^n$$

$$(Txy)_k := \sum_{i,j=1}^{n} T_{ijk} x_i y_j \qquad \forall k = 1, \dots, n$$

$$Tx^{\alpha}x^{\alpha} = \lambda x$$

α-Eigenvectors: Motivation

$T \in \mathbb{R}_{++}^{n \times n \times n}$, $\alpha \in \mathbb{R}$, $p \in (1, \infty)$

$$Tx^{\alpha}x^{\alpha} = \lambda\, x \qquad\qquad x \in \mathbb{S}_{++}^{n}$$

- ℓ^p-eigenvalue problem with $\alpha = \frac{1}{p-1}$

  [Qi (2005), Lim (2005)]

  - H-eigenvalue problem with $\alpha = \frac{1}{2}$
  - Z-eigenvalue problem with $\alpha = 1$
- Polynomial operator norm: $Q \in \mathbb{R}_{++}^{n \times n \times n}, p \in (1, \infty)$

$$\max_{x \neq 0} \frac{\|Qxx\|_{p'}}{\|x\|_p^2} = \max_{x \neq 0} \frac{\langle Qxx, x\rangle}{\|x\|_p^3}$$

- Application in Hypergraph matching with $\alpha = 1$ [Nguyen et al. (2016)]
- NP-hard for $T \in \mathbb{R}_{+}^{n \times n \times n}$ super-symmetric and $\alpha = 1$

  [Hillar, Lim (2013)]

Multilinear Birkhoff-Hopf theorem

Theorem [G., Hein, Tudisco (2018)]

Let $T \in \mathbb{R}^{n\times n\times n}_{++}$, then for every $(x,y),(u,v) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$,

$$\mu(Txy, Tuv) \;\le\; \kappa_1(T)\,\mu(x,u) + \kappa_2(T)\,\mu(y,v)$$

$$\kappa_1(T) \;=\; \max_{i,j,k,i',j',k'} \;\tanh\left[\tfrac{1}{4}\ln\left(\frac{T_{ijk}\,T_{i'j'k'}}{T_{ij'k}\,T_{i'jk'}}\right)\right] < 1$$

$$\kappa_2(T) \;=\; \max_{i,j,k,i',j',k'} \;\tanh\left[\tfrac{1}{4}\ln\left(\frac{T_{ijk}\,T_{i'j'k'}}{T_{ijk'}\,T_{i'j'k}}\right)\right] < 1$$

Multilinear third order eigenvector

$T \in \mathbb{R}_{++}^{n\times n\times n}$

$$Tx x = \lambda\, x \qquad x \in \mathbb{S}_{++}^{n} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$r = \kappa_1(T) + \kappa_2(T) < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution**

- $\exists T \in \mathbb{R}_{++}^{n\times n\times n}$ with $1 < \kappa_1(T) + \kappa_2(T) < 2$ (unlike matrices)

α-Eigenvectors: Perron-Frobenius theorem

$T \in \mathbb{R}_{++}^{n\times n\times n}$, $\alpha \in \mathbb{R}$, $x_0 > 0$

$$Tx^{\alpha}x^{\alpha} = \lambda x \qquad x \in \mathbb{S}_{++}^{n} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$r = (\kappa_1(T) + \kappa_2(T))|\alpha| < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution** $x^\star$ and $\mu(x_{k+1}, x^\star) \le c\, r^k$

$$x_{k+1} = \frac{Tx_k^{\alpha}x_k^{\alpha}}{\| Tx_k^{\alpha}x_k^{\alpha} \|}$$

$$\begin{cases} Tx^{\alpha}y^{\beta} = \lambda x \\ Sx^{\alpha}x^{\alpha} = \sigma y \end{cases}$$

(α, β)-singular vectors: Motivation

$S, T \in \mathbb{R}^{n \times n \times n}_{++}$, $\alpha \in \mathbb{R}$, $p, q \in (1, \infty)$

$$\begin{cases} Tx^{\alpha}y^{\beta} = \lambda\, x \\ Sx^{\alpha}x^{\alpha} = \sigma\, y \end{cases} \qquad x, y \in \mathbb{S}^n_{++} \qquad (\star)$$

- $\ell^{p,q}$-rectangular singular value problem with $\alpha = \frac{1}{p-1}$ and $\beta = \frac{1}{q-1}$ [Chang, Qi, Zhou (2010)]
- Node and layer eigenvector centralities for multiplex networks [Tudisco, Arrigo, G. (2018)]
- Polynomial operator norm: $Q \in \mathbb{R}^{n \times n \times n}_{++}$ and

$$\max_{x \neq 0} \frac{\|Qxy\|_{p'}}{\|x\|_p \|y\|_q} = \max_{x,y \neq 0} \frac{\langle x, Qxy \rangle}{\|x\|_p^2 \|y\|_q}$$

- $(\star)$ is NP-hard for $S = T \in \mathbb{R}^{n \times n \times n}_{+}$, $\alpha = \beta = 1$ [Hillar, Lim (2013)]

Rectangular singular vectors

$R, S, T \in \mathbb{R}^{n\times n\times n}_{++}$

$$\begin{cases} Txy = \lambda\, x \\ Sxx = \sigma\, y \end{cases} \qquad x, y \in \mathbb{S}^n_{++} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$\rho\begin{pmatrix} \kappa_1(T) & \kappa_2(T) \\ \kappa_1(S)+\kappa_2(S) & 0 \end{pmatrix} < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution**

(α, β)-singular vectors: Perron-Frobenius theorem

$S, T \in \mathbb{R}^{n\times n\times n}_{++}$, $\alpha, \beta \in \mathbb{R}$, $x_0, y_0 \in \mathbb{R}^n_{++}$

$$\begin{cases} Tx^\alpha y^\beta & = & \lambda x \\ Sx^\alpha x^\alpha & = & \sigma\, y \end{cases} \qquad (x, y) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$r = \rho\Big(\begin{pmatrix} \kappa_1(T) & \kappa_2(T) \\ \kappa_1(S) + \kappa_2(S) & 0 \end{pmatrix}\begin{pmatrix} |\alpha| & 0 \\ 0 & |\beta| \end{pmatrix}\Big) < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution** $(x^\star, y^\star)$ and
$\mu(x_k, x^\star) + \mu(y_k, y^\star) \le c\, r^k$

$$(x_{k+1}, y_{k+1}) = \Big(\frac{Tx_k^\alpha x_k^\beta}{\| Tx_k^\alpha x_k^\beta \|}, \frac{Tx_k^\alpha x_k^\beta}{\| Tx_k^\alpha x_k^\beta \|}\Big)$$

$$\begin{cases} Ty^{\beta}z^{\gamma} = \lambda x \\ Sx^{\alpha}z^{\gamma} = \sigma y \\ Rx^{\alpha}y^{\beta} = \theta z \end{cases}$$

(α, β, γ)-singular vectors: Motivation

$R, S, T \in \mathbb{R}^{n\times n\times n}_{++}$, $\alpha, \beta, \gamma \in \mathbb{R}$, $p, q, r \in (1, \infty)$

$$\begin{cases} Ty^{\beta}z^{\gamma} = \lambda x \\ Sx^{\alpha}z^{\gamma} = \sigma y \\ Rx^{\alpha}y^{\beta} = \theta z \end{cases} \qquad x, y, z \in \mathbb{S}^n_{++} \qquad (\star)$$

- $\ell^{p,q,r}$-singular value problem with $\alpha = \frac{1}{p-1}, \beta = \frac{1}{q-1}, \gamma = \frac{1}{r-1}$ [Lim (2005)]
- Tensor spectral norm: $Q \in \mathbb{R}^{n\times n\times n}_{++}$

$$\max_{x\neq 0} \frac{\|Qxy\|_{r'}}{\|x\|_p\|y\|_q} = \max_{x,y,z\neq 0} \frac{\langle Qxy, z\rangle}{\|x\|_p\|y\|_q\|z\|_r}$$

- Application in optimal transport with $\alpha = \beta = \gamma = -1$ [Benamou et al. (2015)]
- NP-hard for $R = S = T \in \mathbb{R}^{n\times n\times n}_{+}$, $\alpha = \beta = \gamma = 1$ [Hillar,Lim (2013)]

(α, β, γ)-singular vectors: Perron-Frobenius theorem

$R, S, T \in \mathbb{R}^{n\times n\times n}_{++}$, $\alpha, \beta, \gamma \in \mathbb{R}$, $x_0, y_0, z_0 > 0$

$$\begin{cases} Ty^{\beta}z^{\gamma} = \lambda\, x \\ Sx^{\alpha}z^{\gamma} = \sigma y \\ Rx^{\alpha}y^{\beta} = \theta z \end{cases} \qquad x, y, z \in \mathbb{S}^n_{++} \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

$$\rho\left(\begin{pmatrix} 0 & \kappa_1(T) & \kappa_2(T) \\ \kappa_1(S) & 0 & \kappa_2(S) \\ \kappa_1(R) & \kappa_2(R) & 0 \end{pmatrix} \begin{pmatrix} |\alpha| & 0 & 0 \\ 0 & |\beta| & 0 \\ 0 & 0 & |\gamma| \end{pmatrix} \right) < 1$$

$\Longrightarrow$ $(\star)$ **has a unique solution** $(x^\star, y^\star)$ and
$\mu(x_{k+1}, x^\star) + \mu(y_{k+1}, y^\star) + \mu(z_{k+1}, z^\star) \leq c\, r^k$

$$(x_{k+1}, y_{k+1}, z_{k+1}) = \left(\frac{Ty_k^{\beta} z_k^{\gamma}}{\| Ty_k^{\beta} z_k^{\gamma}\|}, \frac{Sx_k^{\alpha} z_k^{\gamma}}{\| Sx_k^{\alpha} z_k^{\gamma}\|}, \frac{Rx_k^{\alpha} y_k^{\beta}}{\| Rx_k^{\alpha} y_k^{\beta}\|} \right)$$

Tensor norms example

$Q \in \mathbb{R}^{n\times n\times n}_{++}$ super-symmetric, $p \in (1,\infty)$

$$\|Q\|_p = \max_{x\neq 0} \frac{\langle Qxx, x\rangle}{\|x\|_p^3} = \max_{x,y\neq 0} \frac{\langle Qxy, y\rangle}{\|x\|_p\|y\|_p^2} = \max_{x,y,z\neq 0} \frac{\langle Qxy, z\rangle}{\|x\|_p\|y\|_p\|z\|_p}$$

Corollary [G., Hein, Tudisco (2018)]

If $p > 2\kappa_1(Q)$, then $\|Q\|_p$ can be computed by power method (with linear convergence rate).

$Q \in (5,6)^{n\times n\times n}$

Classical condition ([1], [2], [3])	$\Longrightarrow$	$p \geq 3$
$\rho(Q) < 1$	$\Longrightarrow$	$p \geq 1.2$

1: Lim, (2005), 2: Friedland, Gaubert, Han, (2010), 3: G., Tudisco, Hein (2017)

Further generalizations

Tensors of any size and order

$T \in \mathbb{R}_{++}^{n_1 \times n_2 \times \ldots \times n_s}$, $q, p_1, \ldots, p_m \in (1, \infty)$

$$\max_{x_1,\ldots,x_m \neq 0} \frac{\| T \overbrace{x_1 \cdots x_1}^{\nu_1 \text{ times}} \overbrace{x_2 \cdots x_2}^{\nu_2 \text{ times}} \cdots \overbrace{x_m \cdots x_m}^{\nu_m \text{ times}} \|_q}{\|x_1\|_{p_1}^{\nu_1} \, \|x_2\|_{p_2}^{\nu_2} \, \cdots \, \|x_m\|_{p_m}^{\nu_m}}$$

Mode-j Birkhoff contraction ratio

$$\kappa_j(T) = \max_{\substack{i_1,\ldots,i_s \\ i'_1,\ldots,i'_s}} \tanh\left[\frac{1}{4} \ln\left(\frac{T_{i_1,\ldots,i_j,\ldots,i_s} \, T_{i'_1,\ldots,i'_j,\ldots,i'_s}}{T_{i_1,\ldots,i'_j,\ldots,i_s} \, T_{i'_1,\ldots,i_j,\ldots,i'_s}}\right)\right] \qquad \forall j$$

Mixed orders and powers

$R \in \mathbb{R}_{++}^{n\times n\times n}, S \in \mathbb{R}_{++}^{n\times n\times n\times n}, T \in \mathbb{R}_{++}^{n\times n\times n\times n\times n}$

$$\begin{cases} Txyyz = \lambda x^4 \\ Sxyz = \sigma y^3 \\ Rxz = \theta z^2 \end{cases} \qquad x, y, z \in \mathbb{S}_{++}^n$$

$P, Q \in \mathbb{R}_{++}^{n\times n\times n\times n}$, $\alpha_{i,j}, \gamma_i \in \mathbb{R}$

$$\begin{cases} Qx^{\alpha_{1,1}} y^{\alpha_{1,2}} y^{\alpha_{1,3}} = \lambda x^{\gamma_1} \\ Px^{\alpha_{2,1}} x^{\alpha_{2,2}} y^{\alpha_{2,3}} = \sigma y^{\gamma_2} \end{cases} \qquad x, y \in \mathbb{S}_{++}^n$$

Nonnegative tensors and eigenvectors

$S, T \in \mathbb{R}_+^{n\times n\times\ldots\times n}$ of order d and $\alpha, \beta > d-1$

$$\begin{cases} Tyy\cdots y = \lambda x^\alpha \\ Sxx\cdots x = \sigma y^\beta \end{cases} \qquad x, y \in \mathbb{S}_+^n \qquad (\star)$$

Theorem [G., Hein, Tudisco (2018)]

If $S_{i,i,\ldots,i} T_{i,i,\ldots,i} > 0$ for $i = 1, \ldots, n$, then there exists a unique positive solution to $(\star)$ and the power method converges to it with linear rate. Furthermore this solution is "maximal".

Multi-homogeneous mappings and monotonic norms

$A, B \in \mathbb{R}^{n\times n}_{++}$

$$\begin{cases} \big((Ay)^{1/3} \circ (Bz)^{1/3} \big)^2 = \lambda x \\ \max\{Ax, Bx\} = \sigma y \\ \min\{Ay, By\} = \theta z \end{cases} \qquad\qquad x, y, z \in \mathbb{S}^n_{++}$$

$T \in \mathbb{R}^{n\times n\times n}_{++}$, $p_i, q_i, r_i \in (1, \infty)$

$$\max_{\substack{(a,b)\neq 0 \\ u+v\neq 0 \\ z\neq 0}} \frac{\|T(u+v)\binom{a}{b}\|_{p_1} + \|T(u+v)\binom{a}{b}\|_{p_2}}{(\|u\|_{q_1} + \|v\|_{q_2})\sqrt{\|a\|_{r_1}^2 + \|b\|_{r_2}^2}}$$

Non-convex optimization

$\Phi \in C^2(\mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \times \ldots \times \mathbb{R}_+^{n_d}, \mathbb{R})$, $\nabla\Phi > 0$

$$\max_{\substack{x_1,\ldots,x_d \geq 0 \\ \|x_1\|_{p_1} = \cdots = \|x_d\|_{p_d} = 1}} \Phi(x_1, \ldots, x_d)$$

- Application to globally optimal training of generalized polynomial neural networks [G., Nguyen, Hein (2016)]

Parameters: $A_i \in \mathbb{R}^{n\times n}$ orthogonal matrices for $i = 1, \dots, 123$
Variables:

- $g \in C^2([0,1], \mathbb{R})$ strongly convex and $g(0) = g'(0) = 0$
- $M \in \mathbb{R}^{n\times n}$ positive definite
- $(x_m)_{m=1}^{\infty} \subset \mathbb{R}_{++}$ such that $\lim_{m\to\infty} x_m = \xi > 0$

$$\begin{cases} M = \displaystyle\sum_{k=1}^{123} \sqrt{x_k} A_k M^{1/3} A_k^\top & \\ g''(t) = \mathrm{Tr}(M^{-1/2}) \displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\,t)+2}{k^2 \sqrt[3]{x_k}} & \forall t \in [0,1] \\ x_m^5 = \displaystyle\sum_{k=1}^{i+1} \frac{1}{k!} g'\Big(\frac{k-1}{k+1}\Big) \int_{[0,\frac{1}{k}]^n} \langle \gamma, M\gamma \rangle \, d\gamma & \forall m \in \mathbb{N} \end{cases}$$

Thank you for your attention

Presented results can be found in:

- *The Perron-Frobenius theorem for multi-homogeneous mappings*, A. Gautier, F. Tudisco, M. Hein, (2018), arXiv:1801.05034
- *A unifying Perron-Frobenius theorem for nonnegative tensors via multi-homogeneous maps*, A Gautier, F Tudisco, M Hein, (2018), arXiv:1801.04215
- *The contractivity of cone-preserving multilinear mappings* , A. Gautier, F. Tudisco, M. Hein, (2018), arXiv:1808.04180
- *Computing general matrix norms and applications to log-Sobolev constant approximation*, A. Gautier, F. Tudisco, M. Hein, (2018), to appear

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