Local p-adic analysis

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1 Ultrametric spaces

We recall the very basic notions.

Definition. *A metric space* (X,d) *is called ultrametric if the strict triangle inequality*

$$d(x,z) \le \max(d(x,y), d(y,z)) \quad \textit{for any } x,y,z \in X$$

is satisfied.

Remark. *i. If* (X,d) *is ultrametric then* $(Y, d\,|Y \times Y)$*, for any subset* $Y \subseteq X$*, is ultrametric as well.*

ii. If $(X_1,d_1),\ldots,(X_m,d_m)$ *are ultrametric spaces then the cartesian product* $X_1 \times \ldots \times X_m$ *is ultrametric with respect to*

$$d((x_1,\ldots,x_m),(y_1,\ldots,y_m)) := \max(d_1(x_1,y_1),\ldots,d_m(x_m,y_m)) \ .$$

Let (X,d) be an ultrametric space.

Lemma 1.1. *For any three points* $x,y,z \in X$ *such that* $d(x,y) \neq d(y,z)$ *we have*

$$d(x,z) = \max(d(x,y), d(y,z)) \ .$$

Proof. Suppose that $d(x,y) < d(y,z)$. Then

$$d(x,y) < d(y,z) \le \max(d(y,x), d(x,z)) = \max(d(x,y), d(x,z)) = d(x,z) \ .$$

Hence $d(x,y) < d(y,z) \le d(x,z)$ and then

$$d(x,z) \le \max(d(x,y), d(y,z)) \le d(x,z) \ .$$

□

Let $a \in X$ be a point and $\varepsilon > 0$ be a real number. We call

$$B_\varepsilon(a) := \{x \in X : d(a,x) \le \varepsilon\} \quad \text{and} \quad B_\varepsilon^-(a) := \{x \in X : d(a,x) < \varepsilon\}$$

the *closed* and the *open ball*, respectively, around a of radius ε. But be careful!

Lemma 1.2. *i. Every ball is open and closed in* X*.*

ii. For $b \in B_\varepsilon(a)$*, resp.* $b \in B_\varepsilon^-(a)$*, we have* $B_\varepsilon(b) = B_\varepsilon(a)$*, resp.* $B_\varepsilon^-(b) = B_\varepsilon^-(a)$*.*

Proof. Observe that the open, resp. closed, balls are the equivalence classes of the equivalence relation $x \sim y$, resp. $x \approx y$, on X defined by $d(x, y) < \varepsilon$, resp. by $d(x, y) \le \varepsilon$. $\square$

So any point of a ball can serve as its midpoint.

Exercise. *The radius of a ball is not well determined.*

Corollary 1.3. *For any two balls B and B' in X such that $B \cap B' \neq \emptyset$ we have $B \subseteq B'$ or $B' \subseteq B$.*

Remark. *If the ultrametric space X is connected then it is empty or consists of one point.*

Lemma 1.4. *Let $U \subseteq X$ be an open subsets and let $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ be a strictly descending sequence of positive real numbers which converges to zero; then any open covering of U can be refined into a DECOMPOSITION of U into balls of the form $B_{\varepsilon_i}(a)$.*

As usual the metric space X is called *complete* if every Cauchy sequence in X is convergent.

Lemma 1.5. *A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a Cauchy sequence if and only if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

There is a stronger concept than completeness. For a subset $A \subseteq X$ call

$$d(A) := \sup\{d(x, y) : x, y \in A\}$$

the *diameter* of A.

Corollary 1.6. *Let $B \subseteq X$ be a ball with $\varepsilon := d(B) > 0$ and pick any point $a \in B$; we then have $B = B^-_\varepsilon(a)$ or $B = B_\varepsilon(a)$.*

Consider now a descending sequence of balls

$$B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$$

in X. Suppose that X is complete and that $\lim_{n\to\infty} d(B_n) = 0$. We pick points $x_n \in B_n$ and obtain the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$. Since each B_n is closed we must have $x := \lim_{n\to\infty} x_n \in B_n$ and therefore $x \in \bigcap_n B_n$. Hence

$$\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset \ .$$

But without the condition on the diameters the intersection $\bigcap_n B_n$ can be **empty** (see the next section for an important example). This motivates the following definition.

Definition. *The ultrametric space* (X, d) *is called spherically complete if any descending sequence of balls* $B_1 \supseteq B_2 \supseteq \dots$ *in* X *has a nonempty intersection.*

Lemma 1.7. *i. If* X *is spherically complete then it is complete.*

ii. Suppose that X *is complete; if* 0 *is the only accumulation point of the set* $d(X \times X) \subseteq \mathbb{R}_+$ *of values of the metric* d *then* X *is spherically complete.*

Lemma 1.8. *Suppose that* X *is spherically complete; for any family* $(B_i)_{i \in I}$ *of closed balls in* X *such that* $B_i \cap B_j \neq \emptyset$ *for any* $i, j \in I$ *we then have* $\bigcap_{i \in I} B_i \neq \emptyset$.

2 Nonarchimedean fields

Let K be any field.

Definition. *A nonarchimedean absolute value on* K *is a function*

$$| \ | : K \longrightarrow \mathbb{R}$$

which satisfies:

(i) $|a| \geq 0$,

(ii) $|a| = 0$ *if and only if* $a = 0$,

(iii) $|ab| = |a| \cdot |b|$,

(iv) $|a + b| \leq \max(|a|, |b|)$.

Exercise. *i.* $|n \cdot 1| \leq 1$ *for any* $n \in \mathbb{Z}$.

ii. $| \ | : K^\times \longrightarrow \mathbb{R}_+^\times$ *is a homomorphism of groups; in particular,* $|1| = |-1| = 1$.

iii. K *is an ultrametric space with respect to the metric* $d(a, b) := |b - a|$*; in particular, we have* $|a + b| = \max(|a|, |b|)$ *whenever* $|a| \neq |b|$.

iv. Addition and multiplication on the ultrametric space K *are continuous maps.*

Definition. *A nonarchimedean field* $(K, | \ |)$ *is a field* K *equipped with a nonarchimedean absolute value* $| \ |$ *such that:*

(i) $|\ |$ *is non-trivial, i. e., there is an* $a \in K$ *with* $|a| \neq 0, 1$,

(ii) K *is complete with respect to the metric* $d(a, b) := |b - a|$.

Examples. – *Fix a prime number* p. *Then*

$$|a|_p := p^{-r} \quad \text{if } a = p^r \tfrac{m}{n} \text{ with } r, m, n \in \mathbb{Z} \text{ and } p \nmid mn$$

is a nonarchimedean absolute value on the field $\mathbb{Q}$ *of rational numbers. The corresponding completion* $\mathbb{Q}_p$ *is called the field of p-adic numbers. Note that* $|\mathbb{Q}_p|_p = p^{\mathbb{Z}} \cup \{0\}$. *Hence* $\mathbb{Q}_p$ *is spherically complete by Lemma 1.7.ii.*

– *Let* $K/\mathbb{Q}_p$ *be any finite extension of fields. Then*

$$|a| := \sqrt[[K:\mathbb{Q}_p]]{|Norm_{K/\mathbb{Q}_p}(a)|_p}$$

is the unique extension of $|\ |_p$ *to a nonarchimedean absolute value on* K. *The corresponding ultrametric space* K *is complete and spherically complete and, in fact, locally compact. (See [Ser] Chap. II §§1-2.)*

– *The algebraic closure* $\mathbb{Q}_p^{alg}$ *of* $\mathbb{Q}_p$ *is not complete. Its completion* $\mathbb{C}_p$ *is algebraically closed but not spherically complete (see [Sch] §17 and Cor. 20.6).*

In the following we fix a nonarchimedean field $(K, |\ |)$. By the strict triangle inequality the closed unit ball

$$o_K := B_1(0)$$

is a subring of K, called the *ring of integers* in K, and the open unit ball

$$\mathfrak{m}_K := B_1^-(0)$$

is an ideal in o_K. Because of $o_K^\times = o_K \setminus \mathfrak{m}_K$ this ideal $\mathfrak{m}_K$ is the only maximal ideal of o_K. The field $o_K/\mathfrak{m}_K$ is called the *residue class field* of K.

Exercise 2.1. *i. If the residue class field* $o_K/\mathfrak{m}_K$ *has characteristic zero then* K *has characteristic zero as well and we have* $|a| = 1$ *for any nonzero* $a \in \mathbb{Q} \subseteq K$.

ii. If K *has characteristic zero but* $o_K/\mathfrak{m}_K$ *has characteristic* $p > 0$ *then we have*

$$|a| = |a|_p^{-\frac{\log|p|}{\log p}} \qquad \text{for any } a \in \mathbb{Q} \subseteq K;$$

in particular, K *contains* $\mathbb{Q}_p$.

A nonarchimedean field K as in the second part of Exercise 2.1 is called a *p-adic field*.

Lemma 2.2. *If K is p-adic then we have*

$$|n| \geq |n!| \geq |p|^{\frac{n-1}{p-1}} \qquad \textit{for any } n \in \mathbb{N}.$$

Proof. We may obviously assume that $K = \mathbb{Q}_p$. Then the reader should do this as an exercise but also may consult [B-LL] Chap. II §8.1 Lemma 1. □

Now let V be any K-vector space.

Definition. *A (nonarchimedean) norm on V is a function $\| \ \| : V \longrightarrow \mathbb{R}$ such that for any $v, w \in V$ and any $a \in K$ we have:*

(i) $\|av\| = |a| \cdot \|v\|$,

(ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$,

(iii) if $\|v\| = 0$ *then* $v = 0$.

Moreover, V is called normed if it is equipped with a norm.

Exercise. *i.* $\|v\| \geq 0$ *for any* $v \in V$ *and* $\|0\| = 0$.

ii. V is an ultrametric space with respect to the metric $d(v, w) := \|w - v\|$; in particular, we have $\|v + w\| = \max(\|v\|, \|w\|)$ whenever $\|v\| \neq \|w\|$.

iii. Addition $V \times V \xrightarrow{+} V$ and scalar multiplication $K \times V \longrightarrow V$ are continuous.

Lemma 2.3. *Let $(V_1, \| \ \|_1)$ and $(V_2, \| \ \|_2)$ let two normed K-vector spaces; a linear map $f : V_1 \longrightarrow V_2$ is continuous if and only if there is a constant $c > 0$ such that*

$$\|f(v)\|_2 \leq c \cdot \|v\|_1 \quad \textit{for any } v \in V_1 \ .$$

Definition. *The normed K-vector space $(V, \| \ \|)$ is called a K-Banach space if V is complete with respect to the metric $d(v, w) := \|w - v\|$.*

Examples. *1) K^n with the norm $\|(a_1, \ldots, a_n)\| := \max_{1 \leq i \leq n} |a_i|$ is a K-Banach space.*

2) *Let I be a fixed but arbitrary index set. A family $(a_i)_{i \in I}$ of elements in K is called bounded if there is a $c > 0$ such that $|a_i| \le c$ for any $i \in I$. The set*

$$\ell^\infty(I) := \text{ set of all bounded families } (a_i)_{i \in I} \text{ in } K$$

with componentwise addition and scalar multiplication and with the norm

$$\|(a_i)_i\|_\infty := \sup_{i \in I} |a_i|$$

is a K-Banach space.

3) *With I as above let*

$$c_0(I) := \{(a_i)_{i \in I} \in \ell^\infty(I) : \text{for any } \varepsilon > 0 \text{ we have } |a_i| \ge \varepsilon \\ \text{for at most finitely many } i \in I\}.$$

It is a closed vector subspace of $\ell^\infty(I)$ and hence a K-Banach. Moreover, for $(a_i)_i \in c_0(I)$ we have

$$\|(a_i)_i\|_\infty = \max_{i \in I} |a_i| \ .$$

Remark. *Any K-Banach space $(V, \| \ \|)$ over a finite extension $K/\mathbb{Q}_p$ which satisfies $\|V\| \subseteq |K|$ is isometric to some K-Banach space $(c_0(I), \| \ \|_\infty)$; moreover, all such I have the same cardinality.*

Proof. Compare [NFA] Remark 10.2 and Lemma 10.3. □

Let V and W be two normed K-vector spaces. Obviously

$$\mathcal{L}(V, W) := \{f \in \mathrm{Hom}_K(V, W) : f \text{ is continuous}\}$$

is a vector subspace of $\mathrm{Hom}_K(V, W)$. By Lemma 2.3 the *operator norm*

$$\|f\| := \sup\left\{\frac{\|f(v)\|}{\|v\|} : v \in V, v \neq 0\right\} = \sup\left\{\frac{\|f(v)\|}{\|v\|} : v \in V, 0 < \|v\| \le 1\right\}$$

is well defined for any $f \in \mathcal{L}(V, W)$. (Unless it causes confusion all occurring norms will be denoted by $\| \ \|$.)

Proposition 2.4. *i. $\mathcal{L}(V, W)$ with the operator norm is a normed K-vector space.*

ii. If W is a K-Banach space then so, too, is $\mathcal{L}(V, W)$.

In particular,

$$V' := \mathcal{L}(V, K)$$

always is a K-Banach space. It is called the *dual space* to V.

Lemma 2.5. *Let I be an index set; for any $j \in I$ let $1_j \in c_0(I)$ denote the family $(a_i)_{i\in I}$ with $a_i = 0$ for $i \neq j$ and $a_j = 1$; then*

$$\begin{aligned} c_0(I)' &\xrightarrow{\cong} \ell^\infty(I) \\ \ell &\longmapsto (\ell(1_i))_{i\in I} \end{aligned}$$

is an isometric linear isomorphism.

Here is a **warning**.

Proposition 2.6. *Suppose that K is not spherically complete; then*

$$\big(\ell^\infty(\mathbb{N})/c_0(\mathbb{N})\big)' = \{0\} \ .$$

Proof. [PGS] Thm. 4.1.12. □

Throughout the further text $(K, |\ |)$ is a fixed nonarchimedean field.

3 Convergent series

We briefly collect the most basic facts about convergent series in Banach spaces. Let $(V, \|\ \|)$ be a K-Banach space.

Lemma 3.1. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V; we then have:*

i. The series $\sum_{n=1}^\infty v_n$ is convergent if and only if $\lim_{n\to\infty} v_n = 0$;

ii. if the limit $v := \lim_{n\to\infty} v_n$ exists in V and is nonzero then $\|v_n\| = \|v\|$ for all but finitely many $n \in \mathbb{N}$;

iii. let $\sigma : \mathbb{N} \to \mathbb{N}$ be any bijection and suppose that the series $v = \sum_{n=1}^\infty v_n$ is convergent in V; then the series $\sum_{n=1}^\infty v_{\sigma(n)}$ is convergent as well with the same limit v.

The following identities between convergent series are obvious:

- $\sum_{n=1}^\infty av_n = a \cdot \sum_{n=1}^\infty v_n$ for any $a \in K$.
- $(\sum_{n=1}^\infty v_n) + (\sum_{n=1}^\infty w_n) = \sum_{n=1}^\infty (v_n + w_n)$.

Lemma 3.2. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} v_n$ *be convergent series in* K *and* V*, respectively; then the series* $\sum_{n=1}^{\infty} w_n$ *with* $w_n := \sum_{\ell+m=n} a_\ell v_m$ *is convergent, and*

$$\sum_{n=1}^{\infty} w_n = \Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{n=1}^{\infty} v_n\Big) .$$

Analogous assertions hold true for series $\sum_{n_1,\ldots,n_r=1}^{\infty} v_{n_1,\ldots,n_r}$ indexed by multi-indices in $\mathbb{N} \times \ldots \times \mathbb{N}$. But we point out the following additional fact.

Lemma 3.3. *Let* $(v_{m,n})_{m,n\in\mathbb{N}}$ *be a double sequence in* V *such that*

$$\lim_{m+n\to\infty} v_{m,n} = 0 ;$$

we then have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{m,n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} v_{m,n}$$

which, in particular, means that all series involved are convergent.

4 Differentiability

Let V and W be two normed K-vector spaces, let $U \subseteq V$ be an open subset, and let $f : U \longrightarrow W$ be some map.

Definition. *The map* f *is called differentiable in the point* $v_0 \in U$ *if there exists a continuous linear map*

$$D_{v_0} f : V \longrightarrow W$$

such that for any $\varepsilon > 0$ *there is an open neighbourhood* $U_\varepsilon = U_\varepsilon(v_0) \subseteq U$ *of* v_0 *with*

$$\|f(v) - f(v_0) - D_{v_0} f(v - v_0)\| \leq \varepsilon \|v - v_0\| \qquad \text{for any } v \in U_\varepsilon .$$

Exercise. *Check that* $D_{v_0} f$ *is uniquely determined.*

The continuous linear map $D_{v_0} f : V \longrightarrow W$ is called (if it exists) the *derivative* of f in the point $v_0 \in U$. In case $V = K$ we also write $f'(a_0) := D_{a_0} f(1)$.

Remark 4.1. *i. If* f *is differentiable in* v_0 *then it is continuous in* v_0*.*

ii. *(Chain rule) Let* V, W_1, *and* W_2 *be normed* K*-vector spaces,* $U \subseteq V$ *and* $U_1 \subseteq W_1$ *be open subsets, and* $f : U \longrightarrow U_1$ *and* $g : U_1 \longrightarrow W_2$ *be maps; suppose that* f *is differentiable in some* $v_0 \in U$ *and* g *is differentiable in* $f(v_0)$*; then* $g \circ f$ *is differentiable in* v_0 *and*

$$D_{v_0}(g \circ f) = D_{f(v_0)} g \circ D_{v_0} f \ .$$

iii. *A continuous linear map* $u : V \longrightarrow W$ *is differentiable in any* $v_0 \in V$ *and* $D_{v_0} u = u$*; in particular, in the situation of ii. we have*

$$D_{v_0}(u \circ f) = u \circ D_{v_0} f \ .$$

iv. *(Product rule) Let* $V, W_1, \ldots, W_m$, *and* W *be normed* K*-vector spaces, let* $U \subseteq V$ *be an open subset with maps* $f_i : U \longrightarrow W_i$, *and let* $u : W_1 \times \ldots \times W_m \longrightarrow W$ *be a continuous multilinear map; suppose that* $f_1, \ldots, f_m$ *all are differentiable in some point* $v_0 \in U$*; then* $u(f_1, \ldots, f_m) : U \longrightarrow W$ *is differentiable in* v_0 *and*

$$D_{v_0}(u(f_1, \ldots, f_m)) = \sum_{i=1}^{m} u(f_1(v_0), \ldots, D_{v_0} f_i, \ldots, f_m(v_0)) \ .$$

Remark. *Suppose that the vector space* $V = V_1 \oplus \ldots \oplus V_m$ *is the direct sum of finitely many vector spaces* $V_1, \ldots, V_m$*. Then we have the usual notion of the partial derivatives* $D^{(i)}_{v_0} f := D_{v_{0,i}} f_i : V_i \longrightarrow W$ *of* f *in* v_0*. The differentiability of* f *in* v_0 *implies the existence of all partial derivatives together with the identity* $D_{v_0} f = \sum_{i=1}^{m} D^{(i)}_{v_0} f$*.*

Definition. *The map* f *is called strictly differentiable in* $v_0 \in U$ *if there exists a continuous linear map* $D_{v_0} f : V \longrightarrow W$ *such that for any* $\varepsilon > 0$ *there is an open neighbourhood* $U_\varepsilon \subseteq U$ *of* v_0 *with*

$$\|f(v_1) - f(v_2) - D_{v_0} f(v_1 - v_2)\| \leq \varepsilon \|v_1 - v_2\| \qquad \text{for any } v_1, v_2 \in U_\varepsilon \ .$$

Exercise. *Suppose that* f *is strictly differentiable in every point of* U*. Then the map*

$$\begin{aligned} U &\longrightarrow \mathcal{L}(V, W) \\ v &\longmapsto D_v f \end{aligned}$$

is continuous.

Proposition 4.2. *(Local invertibility) Let V and W be K-Banach spaces, $U \subseteq V$ be an open subset, and $f : U \longrightarrow W$ be a map which is strictly differentiable in the point $v_0 \in U$; suppose that the derivative $D_{v_0} f : V \xrightarrow{\cong} W$ is a topological isomorphism; then there are open neighbourhoods $U_0 \subseteq U$ of v_0 and $U_1 \subseteq W$ of $f(v_0)$ such that:*

i. *$f : U_0 \xrightarrow{\simeq} U_1$ is a homeomorphism;*

ii. *the inverse map $g : U_1 \longrightarrow U_0$ is strictly differentiable in $f(v_0)$, and*

$$D_{f(v_0)} g = (D_{v_0} f)^{-1} \ .$$

Concerning the assumption on the derivative in the above proposition we recall the open mapping theorem (cf. [NFA] Cor. 8.7): It says that any continuous linear bijection between K-Banach spaces necessarily is a topological isomorphism. We also point out the trivial fact that any linear map between two finite dimensional K-Banach spaces is continuous.

Remark. *A map $f : X \longrightarrow A$ from some topological space X into some set A is called locally constant if $f^{-1}(a)$ is open (and closed) in X for any $a \in A$. Lemma 1.4 implies that there are plenty of locally constant maps $f : U \longrightarrow W$. They all are strictly differentiable in any $v_0 \in U$ with $D_{v_0} f = 0$.*

5 Power series

Let V be a K-Banach space. A power series $f(X)$ in r variables $X = (X_1, \ldots, X_r)$ with coefficients in V is a formal series

$$f(X) = \sum_{\alpha \in \mathbb{N}_0^r} X^\alpha v_\alpha \qquad \text{with } v_\alpha \in V \ .$$

As usual we abbreviate $X^\alpha := X_1^{\alpha_1} \cdot \ldots \cdot X_r^{\alpha_r}$ and $|\alpha| := \alpha_1 + \ldots + \alpha_r$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^r$.

For any $\varepsilon > 0$ the power series $f(X) = \sum_\alpha X^\alpha v_\alpha$ is called *ε-convergent* if

$$\lim_{|\alpha| \to \infty} \varepsilon^{|\alpha|} \|v_\alpha\| = 0 \ .$$

Remark. *If $f(X)$ is ε-convergent then it also is δ-convergent for any $0 < \delta \leq \varepsilon$.*

The K-vector space

$$\mathcal{F}_\varepsilon(K^r;V) := \text{ all } \varepsilon\text{-convergent power series } f(X) = \sum_{\alpha\in\mathbb{N}_0^r} X^\alpha v_\alpha$$

is normed by

$$\|f\|_\varepsilon := \max_\alpha \varepsilon^{|\alpha|}\|v_\alpha\| \ .$$

Remark. *$\mathcal{F}_\varepsilon(K^r;V)$ is a Banach space whose topology only depends on the topology of V (and not on its specific norm). If $\varepsilon = |c|$ for some $c \in K^\times$ the map*

$$\begin{aligned} c_0(\mathbb{N}_0^r) &\xrightarrow{\cong} \mathcal{F}_{|c|}(K^r;K) \\ (a_\alpha)_\alpha &\longmapsto \sum_\alpha \frac{a_\alpha}{c^{|\alpha|}} X^\alpha \end{aligned}$$

is an isometric linear isomorphism.

Let $B_\varepsilon(0)$ denote the closed ball around zero in K^r of radius ε. We recall that K^r always is equipped with the norm $\|(a_1,\ldots,a_r)\| = \max_{1\le i\le r}|a_i|$. By Lemma 3.1.i. we have the K-linear map

$$\begin{aligned} \mathcal{F}_\varepsilon(K^r;V) &\longrightarrow K\text{-vector space of maps } B_\varepsilon(0) \longrightarrow V \\ f(X) = \sum_\alpha X^\alpha v_\alpha &\longmapsto \tilde{f}(x) := \sum_\alpha x^\alpha v_\alpha \ . \end{aligned}$$

Remark 5.1. *For any $x \in B_\varepsilon(0)$ the linear evaluation map*

$$\begin{aligned} \mathcal{F}_\varepsilon(K^r;V) &\longrightarrow V \\ f &\longmapsto \tilde{f}(x) \end{aligned}$$

is continuous of operator norm ≤ 1.

Proof. We have

$$\|\tilde{f}(x)\| = \|\sum_\alpha x^\alpha v_\alpha\| \le \max_\alpha \varepsilon^{|\alpha|}\|v_\alpha\| = \|f\|_\varepsilon \ .$$

□

The following properties are established by straightforward estimates.

Proposition 5.2. *Let $u : V_1 \times V_2 \longrightarrow V$ be a continuous bilinear map between K-Banach spaces; then*

$$\begin{aligned} U : \mathcal{F}_\varepsilon(K^r; V_1) \times \mathcal{F}_\varepsilon(K^r; V_2) &\longrightarrow \mathcal{F}_\varepsilon(K^r; V) \\ \big(\sum_\alpha X^\alpha v_\alpha, \sum_\alpha X^\alpha w_\alpha\big) &\longmapsto \sum_\alpha X^\alpha \big(\sum_{\beta+\gamma=\alpha} u(v_\beta, w_\gamma)\big) \end{aligned}$$

is a continuous bilinear map satisfying

$$U(f,g)^\sim(x) = u(\tilde{f}(x), \tilde{g}(x)) \qquad \text{for any } x \in B_\varepsilon(0)$$

and any $f \in \mathcal{F}_\varepsilon(K^r; V_1)$ and $g \in \mathcal{F}_\varepsilon(K^r; V_2)$.

Proposition 5.3. *$\mathcal{F}_\varepsilon(K^r; K)$ is a commutative K-algebra with respect to the multiplication*

$$\big(\sum_\alpha b_\alpha X^\alpha\big)\big(\sum_\alpha c_\alpha X^\alpha\big) := \sum_\alpha \big(\sum_{\beta+\gamma} b_\beta c_\gamma\big) X^\alpha \ ;$$

in addition we have

$$(fg)^\sim(x) = \tilde{f}(x)\tilde{g}(x) \qquad \text{for any } x \in B_\varepsilon(0)$$

as well as

$$\|fg\|_\varepsilon = \|f\|_\varepsilon \|g\|_\varepsilon$$

for any $f, g \in \mathcal{F}_\varepsilon(K^r; K)$.

Proposition 5.4. *Let $g \in \mathcal{F}_\delta(K^r; K^n)$ such that $\|g\|_\delta \leq \varepsilon$; then*

$$\begin{aligned} \mathcal{F}_\varepsilon(K^n; V) &\longrightarrow \mathcal{F}_\delta(K^r; V) \\ f(Y) = \sum_\beta Y^\beta v_\beta &\longmapsto f \circ g(X) := \sum_\beta g(X)^\beta v_\beta \end{aligned}$$

is a continuous linear map of operator norm ≤ 1 which satisfies

$$(f \circ g)^\sim(x) = \tilde{f}(\tilde{g}(x)) \qquad \text{for any } x \in B_\delta(0) \subseteq K^r \ .$$

Remark. *For any $g \in \mathcal{F}_\delta(K^r; K^n)$, we have, by Remark 5.1, the inequality*

$$\sup_{x \in B_\delta(0)} \|\tilde{g}(x)\| \leq \|g\|_\delta \ .$$

It is, in general, not an equality. This means that we may have $\tilde{g}(B_\delta(0)) \subseteq B_\varepsilon(0)$ even if $\varepsilon < \|g\|_\delta$. Then, for any $f \in \mathcal{F}_\varepsilon(K^n; V)$, the composite of maps

$\tilde{f} \circ \tilde{g}$ exists but the composite of power series $f \circ g \in \mathcal{F}_\delta(K^r; V)$ may not. An example of such a situation is

$$g(X) := X^p - X \in \mathcal{F}_1(\mathbb{Q}_p; \mathbb{Q}_p) \quad \textit{and} \quad f(Y) := \sum_{n=0}^{\infty} Y^n \in \mathcal{F}_{\frac{1}{p}}(\mathbb{Q}_p; \mathbb{Q}_p) \ .$$

Corollary 5.5. *(Point of expansion) Let $f \in \mathcal{F}_\varepsilon(K^r; V)$ and $y \in B_\varepsilon(0)$; then there exists an $f_y \in \mathcal{F}_\varepsilon(K^r; V)$ such that $\|f_y\|_\varepsilon = \|f\|_\varepsilon$ and*

$$\tilde{f}(x) = \tilde{f}_y(x - y) \qquad \textit{for any } x \in B_\varepsilon(0) = B_\varepsilon(y) \ .$$

Of course, we have the formal partial derivatives $\frac{\partial f}{\partial X_i}(X)$ of $f(X)$. Since $\|\mathbb{N}v_\alpha\| \leq \|v_\alpha\|$ they respect ε-convergence.

Proposition 5.6. *The map $\tilde{f}$ is strictly differentiable in every point $z \in B_\varepsilon(0)$ and satisfies*

$$D_z^{(i)} \tilde{f}(1) = \left(\frac{\partial f}{\partial X_i}\right)^{\sim} (z) \ .$$

Proof. Using Cor. 5.5 and the chain rule one reduces the assertion to the case $z = 0$. Consider the continuous linear map

$$\begin{aligned} D: \qquad K^r &\longrightarrow V \\ (a_1, \ldots, a_r) &\longmapsto \sum_{i=1}^{r} a_i v_{\underline{i}} \ . \end{aligned}$$

Let $\delta > 0$ and choose a $0 < \delta' < \varepsilon$ such that

$$\delta' \frac{\|f\|_\varepsilon}{\varepsilon^2} \leq \delta \ .$$

By induction with respect to $|\alpha|$ one checks that

$$|x^\alpha - y^\alpha| \leq (\delta')^{|\alpha|-1} \|x - y\| \qquad \text{for any } x, y \in B_{\delta'}(0) \ .$$

We now compute

$$\begin{aligned}\|\tilde f(x) - \tilde f(y) - D(x-y)\| &= \|\sum_{|\alpha|\geq 2} (x^\alpha - y^\alpha) v_\alpha\| \\ &\leq \max_{|\alpha|\geq 2} |x^\alpha - y^\alpha| \cdot \|v_\alpha\| \\ &\leq \|f\|_\varepsilon \cdot \max_{|\alpha|\geq 2} \frac{|x^\alpha - y^\alpha|}{\varepsilon^{|\alpha|}} \\ &\leq \|f\|_\varepsilon \cdot \max_{|\alpha|\geq 2} \frac{(\delta')^{|\alpha|-1}}{\varepsilon^{|\alpha|}} \cdot \|x-y\| \\ &= \|f\|_\varepsilon \cdot \frac{\delta'}{\varepsilon^2} \cdot \|x-y\| \\ &\leq \delta \|x-y\|\end{aligned}$$

for any $x, y \in B_{\delta'}(0)$. This proves that $\tilde f$ is strictly differentiable in 0 with $D_0 \tilde f = D$ and hence

$$D_0^{(i)} \tilde f(1) = v_{\underline{i}} = \left(\frac{\partial f}{\partial X_i}\right)^{\sim}(0) \ .$$

□

By Prop. 5.6 the map

$$\begin{aligned}\frac{\partial \tilde f}{\partial x_i} : B_\varepsilon(0) &\longrightarrow V \\ x &\longmapsto D_x^{(i)} \tilde f(1)\end{aligned}$$

is well defined and satisfies

$$\frac{\partial \tilde f}{\partial x_i} = \left(\frac{\partial f}{\partial X_i}\right)^{\sim} \ .$$

Corollary 5.7. *(Taylor expansion) If K has characteristic zero then we have*

$$f(X) = \sum_\alpha X^\alpha \tfrac{1}{\alpha_1! \cdot \ldots \cdot \alpha_r!} \Big(\big(\frac{\partial}{\partial x_1}\big)^{\alpha_1} \ldots \big(\frac{\partial}{\partial x_r}\big)^{\alpha_r} \tilde f \Big)(0) \ .$$

Corollary 5.8. *(Identity theorem for power series) If K has characteristic zero then for any nonzero $f \in \mathcal{F}_\varepsilon(K^r; V)$ there is a point $x \in B_\varepsilon(0)$ such that $\tilde f(x) \neq 0$.*

By more sophisticated techniques (cf. [BGR] 5.1.4 Cor. 5 and subsequent comment) the assumption on the characteristic of K in Cor. 5.8 can be removed. So the map

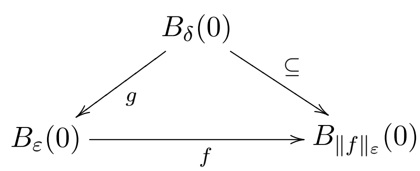
$$\begin{aligned} \mathcal{F}_\varepsilon(K^r; V) &\longrightarrow \text{ strictly differentiable maps } B_\varepsilon(0) \longrightarrow V \\ f &\longmapsto \tilde{f} \end{aligned}$$

always is injective and commutes with all the usual operations as considered above. Therefore we will write very often f for the power series as well as the corresponding map.

Proposition 5.9. *(Invertibility for power series) Let $f(X) \in \mathcal{F}_\varepsilon(K^r; K^r)$ such that $f(0) = 0$, and suppose that $D_0 f$ is bijective; we fix a $0 < \delta < \frac{\varepsilon^2}{\|f\|_\varepsilon \|(D_0 f)^{-1}\|^2}$; then $\delta < \|f\|_\varepsilon$, and there is a uniquely determined $g(Y) \in \mathcal{F}_\delta(K^r; K^r)$ such that*

$$g(0) = 0, \|g\|_\delta < \varepsilon, \ \textit{and } f \circ g(Y) = Y \ ;$$

in particular, the diagram



is commutative.

Finally we note the rather obvious fact.

Proposition 5.10. *Let $u : V \longrightarrow W$ be a continuous linear map between K-Banach spaces; then*

$$\begin{aligned} \mathcal{F}_\varepsilon(K^r; V) &\longrightarrow \mathcal{F}_\varepsilon(K^r; W) \\ f(X) = \sum_\alpha X^\alpha v_\alpha &\longmapsto u \circ f(X) := \sum_\alpha X^\alpha u(v_\alpha) \end{aligned}$$

is a continuous linear map of operator norm $\leq \|u\|$ which satisfies

$$u \circ f(x) = u(f(x)) \qquad \textit{for any } x \in B_\varepsilon(0) \ .$$

6 Locally analytic functions

Let $U \subseteq K^r$ be an open subset and V be a K-Banach space. The key definition in these lectures is the following.

Definition. *A function $f : U \longrightarrow V$ is called locally analytic if for any point $x_0 \in U$ there is a ball $B_\varepsilon(x_0) \subseteq U$ around x_0 and a power series $F \in \mathcal{F}_\varepsilon(K^r; V)$ such that*

$$f(x) = F(x - x_0) \qquad \text{for any } x \in B_\varepsilon(x_0) \ .$$

The set

$$C^{\mathrm{an}}(U, V) := \text{ all locally analytic functions } f : U \longrightarrow V$$

is a K-vector space with respect to pointwise addition and scalar multiplication. The vector space $C^{\mathrm{an}}(U, V)$ carries a natural topology which will be discussed later on.

Example. *By Cor. 5.5 we have $\tilde{F} \in C^{\mathrm{an}}(B_\varepsilon(0), V)$ for any $F \in \mathcal{F}_\varepsilon(K^r; V)$.*

Proposition 6.1. *Suppose that $f : U \longrightarrow V$ is locally analytic; then f is strictly differentiable in every point $x_0 \in U$ and the function $x \longmapsto D_x f$ is locally analytic in $C^{\mathrm{an}}(U, \mathcal{L}(K^r, V))$.*

Proof. Let $F \in \mathcal{F}_\varepsilon(K^r; V)$ such that

$$f(x) = \tilde{F}(x - x_0) \qquad \text{for any } x \in B_\varepsilon(x_0) \ .$$

From Prop. 5.6 and the chain rule we deduce that f is strictly differentiable in every $x \in B_\varepsilon(x_0)$ and

$$D_x f((a_1, \ldots, a_r)) = \sum_{i=1}^{r} a_i D^{(i)}_{x - x_0} \tilde{F}(1) = \sum_{i=1}^{r} a_i \big(\frac{\partial F}{\partial X_i}\big)^{\sim}(x - x_0) \ .$$

Let

$$\frac{\partial F}{\partial X_i}(X) = \sum_{\alpha} X^\alpha v_{i,\alpha} \ .$$

For any multi-index α we introduce the continuous linear map

$$\begin{aligned} L_\alpha : \qquad K^r &\longrightarrow V \\ (a_1, \ldots, a_r) &\longmapsto a_1 v_{1,\alpha} + \ldots + a_r v_{r,\alpha} \ . \end{aligned}$$

Because of $\|L_\alpha\| \leq \max_i \|v_{i,\alpha}\|$ we have

$$G(X) := \sum_\alpha X^\alpha L_\alpha \in \mathcal{F}_\varepsilon(K^r; \mathcal{L}(K^r, V))$$

and

$$D_x f = \tilde{G}(x - x_0) \qquad \text{for any } x \in B_\varepsilon(x_0) \ .$$

□

Remark 6.2. *If K has characteristic zero then, for any function $f : U \longrightarrow V$, the following conditions are equivalent:*

i. f is locally constant;

ii. f is locally analytic with $D_x f = 0$ for any $x \in U$.

Proof. This is an immediate consequence of the Taylor formula in Cor. 5.7. □

We now give a list of more or less obvious properties of locally analytic functions.

1) For any open subset $U' \subseteq U$ we have the linear restriction map

$$\begin{aligned} C^{\mathrm{an}}(U, V) &\longrightarrow C^{\mathrm{an}}(U', V) \\ f &\longmapsto f|U' \ . \end{aligned}$$

2) For any open and closed subset $U' \subseteq U$ we have the linear map

$$\begin{aligned} C^{\mathrm{an}}(U', V) &\longrightarrow C^{\mathrm{an}}(U, V) \\ f &\longmapsto f_!(x) := \begin{cases} f(x) & \text{if } x \in U', \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

called *extension by zero.*

3) If $U = \bigcup_{i \in I} U_i$ is a covering by pairwise disjoint open subsets then

$$\begin{aligned} C^{\mathrm{an}}(U, V) &\cong \prod_{i \in I} C^{\mathrm{an}}(U_i, V) \\ f &\longmapsto (f|U_i)_i \ . \end{aligned}$$

4) For any two K-Banach spaces V and W we have

$$\begin{aligned} C^{\mathrm{an}}(U, V \oplus W) &\cong C^{\mathrm{an}}(U, V) \oplus C^{\mathrm{an}}(U, W) \\ f &\longmapsto (\mathrm{pr}_V \circ f, \mathrm{pr}_W \circ f) \ . \end{aligned}$$

In particular

$$C^{\mathrm{an}}(U, K^n) \cong \prod_{i=1}^{n} C^{\mathrm{an}}(U, K) \ .$$

5) For any continuous bilinear map $u : V_1 \times V_2 \longrightarrow V$ between K-Banach spaces we have the bilinear map

$$\begin{aligned} C^{\mathrm{an}}(U, V_1) \times C^{\mathrm{an}}(U, V_2) &\longrightarrow C^{\mathrm{an}}(U, V) \\ (f, g) &\longmapsto u(f, g) \end{aligned}$$

(cf. Prop. 5.2). In particular, $C^{\mathrm{an}}(U, K)$ is a K-algebra (cf. Prop. 5.3), and $C^{\mathrm{an}}(U, V)$ is a module over $C^{\mathrm{an}}(U, K)$.

6) For any continuous linear map $u : V \longrightarrow W$ between K-Banach spaces we have the linear map

$$\begin{aligned} C^{\mathrm{an}}(U, V) &\longrightarrow C^{\mathrm{an}}(U, W) \\ f &\longmapsto u \circ f \end{aligned}$$

(cf. Prop. 5.10).

Lemma 6.3. *Let $U' \subseteq K^n$ be an open subset and let $g \in C^{\mathrm{an}}(U, K^n)$ such that $g(U) \subseteq U'$; then the map*

$$\begin{aligned} C^{\mathrm{an}}(U', V) &\longrightarrow C^{\mathrm{an}}(U, V) \\ f &\longmapsto f \circ g \end{aligned}$$

is well defined and K-linear.

Proof. Let $x_0 \in U$ and put $y_0 := g(x_0) \in U'$. We choose a ball $B_\varepsilon(y_0) \subseteq U'$ and a power series $F \in \mathcal{F}_\varepsilon(K^n; V)$ such that

$$f(y) = F(y - y_0) \qquad \text{for any } y \in B_\varepsilon(y_0) \ .$$

We also choose a ball $B_\delta(x_0) \subseteq U$ and a power series $G \in \mathcal{F}_\delta(K^r; K^n)$ such that

$$g(x) = G(x - x_0) \qquad \text{for any } x \in B_\delta(x_0) \ .$$

Observing that

$$\|G - G(0)\|_{\delta'} \leq \frac{\delta'}{\delta}\|G - G(0)\|_\delta \qquad \text{for any } 0 < \delta' \leq \delta$$

we may decrease δ so that

$$\|G - y_0\|_\delta = \|G - G(0)\|_\delta \leq \varepsilon$$

(and, in particular, $g(B_\delta(x_0)) \subseteq B_\varepsilon(y_0))$ holds true. It then follows from Prop. 5.4 that $F \circ (G - y_0) \in \mathcal{F}_\delta(K^r; V)$ and

$$\begin{aligned}(F \circ (G - y_0))^\sim(x - x_0) &= F(G(x - x_0) - y_0) \\ &= F(g(x) - y_0) \\ &= f(g(x))\end{aligned}$$

for any $x \in B_\delta(x_0)$. □

The last result can be expressed by saying that the composite of locally analytic functions again is locally analytic.

Proposition 6.4. *(Local invertibility) Let $U \subseteq K^r$ be an open subset and let $f \in C^{\mathrm{an}}(U, K^r)$; suppose that $D_{x_0}f$ is bijective for some $x_0 \in U$; then there are open neighbourhoods $U_0 \subseteq U$ of x_0 and $U_1 \subseteq K^r$ of $f(x_0)$ such that:*

i. $f : U_0 \xrightarrow{\simeq} U_1$ is a homeomorphism;

ii. the inverse map $g : U_1 \longrightarrow U_0$ is locally analytic, i. e., $g \in C^{\mathrm{an}}(U_1, K^r)$.

A map $f : U \longrightarrow U'$ between open subsets $U \subseteq K^r$ and $U' \subseteq K^n$ is called locally analytic if the composite $U \xrightarrow{f} U' \xrightarrow{\subseteq} K^n$ is a locally analytic function.

7 Charts and atlases

We continue to fix the nonarchimedean field $(K, |\ |)$. But from now on we will denote K-Banach spaces by letters like E whereas letters like U and V are reserved for open subsets in a topological space.

Let M be a Hausdorff topological space.

Definition. *i. A chart for M is a triple (U, φ, K^n) consisting of an open subset $U \subseteq M$ and a map $\varphi : U \longrightarrow K^n$ such that:*

(a) $\varphi(U)$ *is open in* K^n,

(b) $\varphi : U \xrightarrow{\simeq} \varphi(U)$ *is a homeomorphism.*

ii. Two charts $(U_1, \varphi_1, K^{n_1})$ *and* $(U_2, \varphi_2, K^{n_2})$ *for* M *are called compatible if both maps*

$$\varphi_1(U_1 \cap U_2) \underset{\varphi_1 \circ \varphi_2^{-1}}{\overset{\varphi_2 \circ \varphi_1^{-1}}{\rightleftarrows}} \varphi_2(U_1 \cap U_2)$$

are locally analytic.

We note that the condition in part ii. of the above definition makes sense since $\varphi_1(U_1 \cap U_2)$ is open in K^{n_i}. If (U, φ, K^n) is a chart then the open subset U is called its *domain of definition* and the integer $n \geq 0$ its *dimension*. Usually we simply write (U, φ) instead of (U, φ, K^n). If x is a point in U then (U, φ) is also called a *chart around* x.

Lemma 7.1. *Let* $(U_i, \varphi_i, K^{n_i})$ *for* $i = 1, 2$ *be two compatible charts for* M*; if* $U_1 \cap U_2 \neq \emptyset$ *then* $n_1 = n_2$.

Proof. Let $x \in U_1 \cap U_2$ and put $x_i := \varphi_i(x)$. We consider the locally analytic maps

$$\varphi_1(U_1 \cap U_2) \underset{g := \varphi_1 \circ \varphi_2^{-1}}{\overset{f := \varphi_2 \circ \varphi_1^{-1}}{\rightleftarrows}} \varphi_2(U_1 \cap U_2) \ .$$

They are differentiable and inverse to each other, and $x_2 = f(x_1)$. Hence, by the chain rule, the derivatives

$$K^{n_1} \underset{D_{x_2} g}{\overset{D_{x_1} f}{\rightleftarrows}} K^{n_2}$$

are linear maps inverse to each other. It follows that $n_1 = n_2$. □

Definition. *i. An atlas for* M *is a set* $\mathcal{A} = \{(U_i, \varphi_i, K^{n_i})\}_{i \in I}$ *of charts for* M *any two of which are compatible and which cover* M *in the sense that* $M = \bigcup_{i \in I} U_i$.

ii. Two atlases $\mathcal{A}$ *and* $\mathcal{B}$ *for* M *are called equivalent if* $\mathcal{A} \cup \mathcal{B}$ *also is an atlas for* M.

iii. An atlas $\mathcal{A}$ *for* M *is called maximal if any equivalent atlas* $\mathcal{B}$ *for* M *satisfies* $\mathcal{B} \subseteq \mathcal{A}$.

Remark 7.2. *i. The equivalence of atlases indeed is an equivalence relation.*

ii. In each equivalence class of atlases there is exactly one maximal atlas.

Lemma 7.3. *If $\mathcal{A}$ is a maximal atlas for M the domains of definition of all the charts in $\mathcal{A}$ form a basis of the topology of M.*

Definition. *An atlas $\mathcal{A}$ for M is called n-dimensional if all the charts in $\mathcal{A}$ with nonempty domain of definition have dimension n.*

Remark 7.4. *Let $\mathcal{A}$ be an n-dimensional atlas for M; then any atlas $\mathcal{B}$ equivalent to $\mathcal{A}$ is n-dimensional as well.*

8 Manifolds

Definition. *A (locally analytic) manifold $(M, \mathcal{A})$ (over K) is a Hausdorff topological space M equipped with a maximal atlas $\mathcal{A}$. The manifold is called n-dimensional (we write $\dim M = n$) if the atlas $\mathcal{A}$ is n-dimensional.*

We usually speak of a manifold M while considering $\mathcal{A}$ as given implicitly. A chart for M will always mean a chart in $\mathcal{A}$.

Example. *K^n will always denote the n-dimensional manifold whose maximal atlas is equivalent to the atlas $\{(U, \subseteq, K^n) : U \subseteq K^n \text{ open}\}$.*

Remark 8.1. *Let (U, φ, K^n) be a chart for the manifold M; if $V \subseteq U$ is an open subset then $(V, \varphi|V, K^n)$ also is a chart for M.*

Let $(M, \mathcal{A})$ be a manifold and $U \subseteq M$ be an open subset. Then

$$\mathcal{A}_U := \{(V, \psi, K^n) \in \mathcal{A} : V \subseteq U\} ,$$

by Lemma 7.3, is an atlas for U. Check that $\mathcal{A}_U$ is maximal. The manifold $(U, \mathcal{A}_U)$ is called an *open submanifold* of $(M, \mathcal{A})$.

Example. *The d-dimensional projective space $\mathbb{P}^d(K) = (K^{d+1} \setminus \{0\})/\sim$ over K is the set of equivalence classes in $K^{d+1} \setminus \{0\}$ for the equivalence relation*

$$(a_1, \ldots, a_{d+1}) \sim (ca_1, \ldots, ca_{d+1}) \text{ for any } c \in K^\times .$$

As usual we write $[a_1 : \ldots : a_{d+1}]$ for the equivalence class of $(a_1, \ldots, a_{d+1})$. With respect to the quotient topology from $K^{d+1} \setminus \{0\}$ the projective space

$\mathbb{P}^d(K)$ is a Hausdorff topological space. For any $1 \leq j \leq d+1$ we have the open subset

$$U_j := \{[a_1 : \ldots : a_{d+1}] \in \mathbb{P}^d(K) : |a_i| \leq |a_j| \text{ for any } 1 \leq i \leq d+1\}$$

together with the homeomorphism

$$\begin{aligned} \varphi_j : \qquad U_j &\xrightarrow{\simeq} B_1(0) \subseteq K^d \\ [a_1 : \ldots : a_{d+1}] &\longmapsto \left(\tfrac{a_1}{a_j}, \ldots, \tfrac{a_{j-1}}{a_j}, \tfrac{a_{j+1}}{a_j}, \ldots, \tfrac{a_{d+1}}{a_j}\right) . \end{aligned}$$

The (U_j, φ_j, K^d) are charts for $\mathbb{P}^d(K)$ such that $\bigcup_j U_j = \mathbb{P}^d(K)$. They are pairwise compatible. For example, for $1 \leq j < k \leq d+1$, check that the composite map

$$f : \{x \in B_1(0) : |x_{k-1}| = 1\} \xrightarrow{\varphi_j^{-1}} U_j \cap U_k \xrightarrow{\varphi_k} \{y \in B_1(0) : |y_j| = 1\},$$

which is given by

$$f(x_1, \ldots, x_d) = \left(\tfrac{x_1}{x_{k-1}}, \ldots, \tfrac{x_{j-1}}{x_{k-1}}, \tfrac{1}{x_{k-1}}, \tfrac{x_j}{x_{k-1}}, \ldots, \tfrac{x_{k-2}}{x_{k-1}}, \tfrac{x_k}{x_{k-1}}, \ldots, \tfrac{x_d}{x_{k-1}}\right) ,$$

is locally analytic. The above charts therefore form a d-dimensional atlas for $\mathbb{P}^d(K)$.

Exercise. *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two manifolds. Then*

$$\mathcal{A} \times \mathcal{B} := \{(U \times V), \varphi \times \psi, K^{m+n}) : (U, \varphi, K^m) \in \mathcal{A}, (V, \psi, K^n) \in \mathcal{B}\}$$

is an atlas for $M \times N$ with the product topology. We call $M \times N$ equipped with the equivalent maximal atlas the product manifold of M and N.

Let M be a manifold and E be a K-Banach space.

Definition. *A function $f : M \longrightarrow E$ is called locally analytic if $f \circ \varphi^{-1} \in C^{\text{an}}(\varphi(U), E)$ for any chart (U, φ) for M.*

Remark 8.2. *i. Every locally analytic function $f : M \longrightarrow E$ is continuous.*

ii. Let $\mathcal{B}$ be any atlas consisting of charts for M; a function $f : M \longrightarrow E$ is locally analytic if and only if $f \circ \varphi^{-1} \in C^{\text{an}}(\varphi(U), E)$ for any $(U, \varphi) \in \mathcal{B}$.

The set

$$C^{\mathrm{an}}(M,E) := \text{ all locally analytic functions } f : M \longrightarrow E$$

is a K-vector space with respect to pointwise addition and scalar multiplication. It is easy to see that a list of properties 1) - 6) completely analogous to the one given in section 6 holds true.

Let now M and N be two manifolds. The following result is immediate.

Lemma 8.3. *For a map $g : M \longrightarrow N$ the following assertions are equivalent:*

i. g is continuous and $\psi \circ g \in C^{\mathrm{an}}(g^{-1}(V), K^n)$ for any chart (V, ψ, K^n) for N;

ii. for any point $x \in M$ there exist a chart (U, φ, K^m) for M around x and a chart (V, ψ, K^n) for N around $g(x)$ such that $g(U) \subseteq V$ and $\psi \circ g \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), K^n)$.

Definition. *A map $g : M \longrightarrow N$ is called locally analytic if the equivalent conditions in Lemma 8.3 are satisfied.*

Lemma 8.4. *i. If $g : M \longrightarrow N$ is a locally analytic map and E is a K-Banach space then*

$$\begin{aligned} C^{\mathrm{an}}(N,E) &\longrightarrow C^{\mathrm{an}}(M,E) \\ f &\longmapsto f \circ g \end{aligned}$$

is a well defined K-linear map.

ii. With $L \xrightarrow{f} M \xrightarrow{g} N$ also $g \circ f : L \longrightarrow N$ is a locally analytic map of manifolds.

Proof. This follows from Lemma 6.3. □

Examples 8.5. *1) For any open submanifold U of M the inclusion map $U \xrightarrow{\subseteq} M$ is locally analytic.*

2) Let $g : M \longrightarrow N$ be a locally analytic map; for any open submanifold $V \subseteq N$ the induced map $g^{-1}(V) \xrightarrow{g} V$ is locally analytic.

3) The two projection maps

$$\mathrm{pr}_1 : M \times N \longrightarrow M \quad \textit{and} \quad \mathrm{pr}_2 : M \times N \longrightarrow N$$

are locally analytic.

4) For any pair of locally analytic maps $g : L \longrightarrow M$ and $f : L \longrightarrow N$ the map

$$\begin{aligned} (g, f) : L &\longrightarrow M \times N \\ x &\longmapsto (g(x), f(x)) \end{aligned}$$

is locally analytic.

We finish this section by mentioning a very useful technical property of manifolds. First let X be an arbitrary Hausdorff topological space. We recall:

- Let $X = \bigcup_{i \in I} U_i$ and $X = \bigcup_{j \in J} V_j$ be two open coverings of X. The second one is called a *refinement* of the first if for any $j \in J$ there is an $i \in I$ such that $V_j \subseteq U_i$.

- An open covering $X = \bigcup_{i \in I} U_i$ of X is called *locally finite* if every point $x \in X$ has an open neighbourhood U_x such that the set $\{i \in I : U_x \cap U_i \neq \emptyset\}$ is finite.

- The space X is called *paracompact*, resp. *strictly paracompact*, if any open covering of X can be refined into an open covering which is locally finite, resp. which consists of pairwise disjoint open subsets.

Remark 8.6. *i. Any ultrametric space X is strictly paracompact.*

ii. Any compact space X is paracompact.

Proof. i. This follows from Lemma 1.4. ii. This is trivial. □

Proposition 8.7. *For a manifold M the following conditions are equivalent:*

i. M is paracompact;

ii. M is strictly paracompact;

iii. the topology of M can be defined by a metric which satisfies the strict triangle inequality.

Corollary 8.8. *Open submanifolds and product manifolds of paracompact manifolds are paracompact.*

9 The tangent space

Let M be a manifold, and fix a point $a \in M$. We consider pairs (c, v) where

- $c = (U, \varphi, K^m)$ is a chart for M around a and
- $v \in K^m$.

Two such pairs (c, v) and (c', v') are called equivalent if we have

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v' \ .$$

It follows from the chain rule that this indeed defines an equivalence relation.

Definition. *A tangent vector of M at the point a is an equivalence class $[c, v]$ of pairs (c, v) as above.*

We define

$$T_a(M) := \text{ set of all tangent vectors of } M \text{ at } a \ .$$

Lemma 9.1. *Let $c = (U, \varphi, K^m)$ and $c' = (U', \varphi', K^m)$ be two charts for M around a; we then have:*

i. The map

$$\begin{aligned} \theta_c : K^m &\xrightarrow{\sim} T_a(M) \\ v &\longmapsto [c, v] \end{aligned}$$

is bijective.

ii. $\theta_{c'}^{-1} \circ \theta_c : K^m \xrightarrow{\cong} K^m$ is a K-linear isomorphism.

Proof. (We recall from Lemma 7.1 that the dimensions of two charts around the same point necessarily coincide.) i. Surjectivity follows from

$$[c'', v''] = [c, D_{\varphi''(a)}(\varphi \circ \varphi''^{-1})(v'')] \ .$$

If $[c, v] = [c, v']$ then $v' = D_{\varphi(a)}(\varphi \circ \varphi^{-1})(v) = v$. This proves the injectivity.

ii. From $[c, v] = [c', D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v)]$ we deduce that

$$\theta_{c'}^{-1} \circ \theta_c = D_{\varphi(a)}(\varphi' \circ \varphi^{-1}) \ .$$

□

The set $T_a(M)$, by Lemma 9.1.i., has precisely one structure of a topological K-vector space such that the map θ_c is a K-linear homeomorphism. Because of Lemma 9.1.ii. this structure is independent of the choice of the chart c around a.

Definition. *The K-vector space $T_a(M)$ is called the tangent space of M at the point a.*

Remark. *The manifold M has dimension m if and only if $\dim_K T_a(M) = m$ for any $a \in M$.*

Let $g : M \longrightarrow N$ be a locally analytic map of manifolds. By Lemma 8.3.ii. we find charts $c = (U, \varphi, K^m)$ for M around a and $\tilde{c} = (V, \psi, K^n)$ for N around $g(a)$ such that $g(U) \subseteq V$. The composite

$$T_a(g) : T_a(M) \xrightarrow{\theta_c^{-1}} K^m \xrightarrow{D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1})} K^n \xrightarrow{\theta_{\tilde{c}}} T_{g(a)}(N)$$

is a continuous K-linear map. We claim that $T_a(g)$ does not depend on the particular choice of charts. Let $c' = (U', \varphi')$ and $\tilde{c}' = (V', \psi')$ be other charts around a and $g(a)$, respectively. Using the identity in the proof of Lemma 9.1.ii. as well as the chain rule we compute

$$\begin{aligned}
&\theta_{\tilde{c}} \circ D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1}) \circ \theta_c^{-1} \\
&\quad = \theta_{\tilde{c}'} \circ D_{\psi(g(a))}(\psi' \circ \psi^{-1}) \circ D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1}) \circ D_{\varphi(a)}(\varphi' \circ \varphi^{-1})^{-1} \circ \theta_{c'}^{-1} \\
&\quad = \theta_{\tilde{c}'} \circ D_{\varphi'(a)}(\psi' \circ g \circ \varphi'^{-1}) \circ \theta_{c'}^{-1} \ .
\end{aligned}$$

Definition. *$T_a(g)$ is called the tangent map of g at the point a.*

Remark. $T_a(\mathrm{id}_M) = \mathrm{id}_{T_a(M)}$.

Lemma 9.2. *For any locally analytic maps of manifolds $L \xrightarrow{f} M \xrightarrow{g} N$ we have*

$$T_a(g \circ f) = T_{f(a)}(g) \circ T_a(f) \qquad \text{for any } a \in L \ .$$

Proof. This is an easy consequence of the chain rule. □

Proposition 9.3. *(Local invertibility) Let $g : M \longrightarrow N$ be a locally analytic map of manifolds, and suppose that $T_a(g) : T_a(M) \xrightarrow{\cong} T_{g(a)}(N)$ is bijective for some $a \in M$; then there are open neighbourhoods $U \subseteq M$ of a and $V \subseteq N$ of $g(a)$ such that g restricts to a locally analytic isomorphism*

$$g : U \xrightarrow{\simeq} V$$

of open submanifolds.

Proof. This is a consequence of Prop. 6.4. □

Exercise. *Let (U, φ, K^m) be a chart for the manifold M; then $\varphi : U \xrightarrow{\simeq} \varphi(U)$ is a locally analytic isomorphism between the open submanifolds U of M and $\varphi(U)$ of K^m.*

Let M be a manifold, E be a K-Banach space, $f \in C^{\mathrm{an}}(M, E)$, and $a \in M$. If $c = (U, \varphi, K^m)$ is a chart for M around a then $f \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), E)$. Hence

$$\begin{aligned} d_a f : T_a(M) &\xrightarrow{\theta_c^{-1}} K^m \xrightarrow{D_{\varphi(a)}(f\circ\varphi^{-1})} E \\ [c, v] &\longmapsto D_{\varphi(a)}(f \circ \varphi^{-1})(v) \end{aligned}$$

is a continuous K-linear map. If $c' = (U', \varphi', K^m)$ is another chart around a then

$$\begin{aligned} D_{\varphi(a)}(f \circ \varphi^{-1}) \circ \theta_c^{-1} &= D_{\varphi(a)}(f \circ \varphi^{-1}) \circ D_{\varphi(a)}(\varphi' \circ \varphi^{-1})^{-1} \circ \theta_{c'}^{-1} \\ &= D_{\varphi'(a)}(f \circ \varphi'^{-1}) \circ \theta_{c'}^{-1} \ . \end{aligned}$$

This shows that $d_a f$ does not depend on the choice of the chart c.

Definition. *$d_a f$ is called the derivative of f in the point a.*

Remark 9.4. *For $E = K^r$ viewed as a manifold and for the chart $c_0 = (K^r, \mathrm{id}, E)$ for E we have*

$$T_a(f) = \theta_{c_0} \circ d_a f \ .$$

Obviously the map

$$\begin{aligned} C^{\mathrm{an}}(M, E) &\longrightarrow \mathcal{L}(T_a(M), E) \\ f &\longmapsto d_a f \end{aligned}$$

is K-linear.

Lemma 9.5. *(Product rule)*

i. Let $u : E_1 \times E_2 \longrightarrow E$ be a continuous bilinear map between K-Banach spaces; if $f_i \in C^{\mathrm{an}}(M, E_i)$ for $i = 1, 2$ then $u(f_1, f_2) \in C^{\mathrm{an}}(M, E)$ and

$$d_a(u(f_1, f_2)) = u(f_1(a), d_a f_2) + u(d_a f_1, f_2(a)) \qquad \text{for any } a \in M \ .$$

ii. For $g \in C^{\mathrm{an}}(M, K)$ and $f \in C^{\mathrm{an}}(M, E)$ we have

$$d_a(gf) = g(a) \cdot d_a f + d_a g \cdot f(a) \qquad \text{for any } a \in M \ .$$

Let $c = (U, \varphi, K^m)$ be a chart for M. On the one hand, by definition, we have $d_a\varphi = \theta_c^{-1}$ for any $a \in U$; in particular

$$d_a\varphi : T_a(M) \xrightarrow{\cong} K^m$$

is a K-linear isomorphism. On the other hand viewing $\varphi = (\varphi_1, \ldots, \varphi_m)$ as a tuple of locally analytic functions $\varphi_i : U \longrightarrow K$ we have

$$d_a\varphi = (d_a\varphi_1, \ldots, d_a\varphi_m) \ .$$

This means that $\{d_a\varphi_i\}_{1\leq i\leq m}$ is a K-basis of the dual vector space $T_a(M)'$. Let

$$\big\{\big(\tfrac{\partial}{\partial\varphi_i}\big)(a)\big\}_{1\leq i\leq m}$$

denote the corresponding dual basis of $T_a(M)$, i. e.,

$$d_a\varphi_i\big(\big(\tfrac{\partial}{\partial\varphi_j}\big)(a)\big) = \delta_{ij} \qquad \text{for any } a \in U$$

where δ_{ij} is the Kronecker symbol. For any $f \in C^{\mathrm{an}}(M, E)$ we define the functions

$$\begin{aligned} \tfrac{\partial f}{\partial\varphi_i} : U &\longrightarrow E \\ a &\longmapsto d_af\big(\big(\tfrac{\partial}{\partial\varphi_i}\big)(a)\big) \ . \end{aligned}$$

Lemma 9.6. $\frac{\partial f}{\partial\varphi_i} \in C^{\mathrm{an}}(U, E)$ *for any* $1 \leq i \leq m$, *and*

$$d_af = \sum_{i=1}^{m} d_a\varphi_i \cdot \tfrac{\partial f}{\partial\varphi_i}(a) \qquad \textit{for any } a \in U \ .$$

Now we define the disjoint union

$$T(M) := \bigcup_{a\in M} T_a(M)$$

together with the projection map

$$\begin{aligned} p_M : T(M) &\longrightarrow M \\ t &\longmapsto a \ \ \text{if} \ \ t \in T_a(M) \ . \end{aligned}$$

Hence $T_a(M) = p_M^{-1}(a)$. We will show that $T(M)$ is naturally a manifold and p_M a locally analytic map of manifolds.

Consider any chart $c = (U, \varphi, K^m)$ for M. By Lemma 9.1.i. the map

$$\tau_c : U \times K^m \xrightarrow{\sim} p_M^{-1}(U)$$
$$(a, v) \longmapsto [c, v] \text{ viewed in } T_a(M)$$

is bijective. Hence the composite

$$\varphi_c : p_M^{-1}(U) \xrightarrow{\tau_c^{-1}} U \times K^m \xrightarrow{\varphi \times \mathrm{id}} K^m \times K^m = K^{2m}$$

is a bijection onto an open subset in K^{2m}. The idea is that

$$c_T := \big(p_M^{-1}(U), \varphi_c, K^{2m}\big)$$

should be a chart for the manifold $T(M)$. Clearly we have

$$T(M) = \bigcup_{c=(U,\varphi)} p_M^{-1}(U) \ .$$

We equip $T(M)$ with the finest topology which makes all composed maps

$$U \times K^m \xrightarrow{\tau_c} p_M^{-1}(U) \xrightarrow{\subseteq} T(M)$$

continuous.

Lemma 9.7. *i. The map $\tau_c : U \times K^m \xrightarrow{\simeq} p_M^{-1}(U)$ is a homeomorphism with respect to the subspace topology induced by $T(M)$ on $p_M^{-1}(U)$.*

ii. The map p_M is continuous.

iii. The topological space $T(M)$ is Hausdorff.

The Lemma 9.7 in particular says that c_T indeed is a chart for $T(M)$. Check that these charts are compatible. We conclude that the set

$$\{c_T : c \text{ a chart for } M\}$$

is an atlas for $T(M)$ and we always view $T(M)$ as a manifold with respect to the equivalent maximal atlas.

Definition. *The manifold $T(M)$ is called the tangent bundle of M.*

Remark. *If M is m-dimensional then $T(M)$ is $2m$-dimensional.*

Lemma 9.8. *The map $p_M : T(M) \longrightarrow M$ is locally analytic.*

Proof. Let $c = (U, \varphi, K^m)$ be a chart for M. It suffices to contemplate the commutative diagram

$$\begin{array}{ccccccc}
T(M) & \xleftarrow{\supseteq} & p_M^{-1}(U) & \longrightarrow & \varphi_c(p_M^{-1}(U)) = \varphi(U) \times K^m & \xrightarrow{\subseteq} & K^{2m} \\
\downarrow{\scriptstyle p_M} & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\
M & \xleftarrow{\supseteq} & U & \xrightarrow{\varphi} & \varphi(U) & \xrightarrow{\subseteq} & K^m \ .
\end{array}$$

□

Let $g : M \longrightarrow N$ be a locally analytic map of manifolds. We define the map

$$T(g) : T(M) \longrightarrow T(N)$$

by

$$T(g)|T_a(M) := T_a(g) \qquad \text{for any } a \in M \ .$$

In particular, the diagram

$$\begin{array}{ccc}
T(M) & \xrightarrow{T(g)} & T(N) \\
{\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\
M & \xrightarrow{g} & N
\end{array}$$

is commutative.

Proposition 9.9. *i. The map $T(g)$ is locally analytic.*

ii. For any locally analytic maps of manifolds $L \xrightarrow{f} M \xrightarrow{g} N$ we have

$$T(g \circ f) = T(g) \circ T(f) \ .$$

Note that the above ii. is a restatement of Lemma 9.2.

Exercise 9.10. *i. For $U \subseteq M$ an open submanifold, $T(\subseteq)$ induces an isomorphism between $T(U)$ and the open submanifold $p_M^{-1}(U)$.*

ii. For any two manifolds M and N the map

$$T(\mathrm{pr}_1) \times T(\mathrm{pr}_2) : T(M \times N) \xrightarrow{\simeq} T(M) \times T(N)$$

is an isomorphism of manifolds.

Now let M be a manifold and E be a K-Banach space. For any $f \in C^{\mathrm{an}}(M, E)$ we define

$$\begin{aligned} df : T(M) &\longrightarrow E \\ t &\longmapsto d_{p_M(t)} f(t) \ . \end{aligned}$$

Lemma 9.11. *We have $df \in C^{\mathrm{an}}(T(M), E)$.*

Lemma 9.12. *Let $g : M \longrightarrow N$ be a locally analytic map of manifolds; for any $f \in C^{\mathrm{an}}(N, E)$ we have*

$$d(f \circ g) = df \circ T(g) \ .$$

Proof. This is a consequence of the chain rule. □

Exercise. *The map*

$$\begin{aligned} d : C^{\mathrm{an}}(M, E) &\longrightarrow C^{\mathrm{an}}(T(M), E) \\ f &\longmapsto df \end{aligned}$$

is K-linear.

Remark 9.13. *If K has characteristic zero then a function $f \in C^{\mathrm{an}}(M, E)$ is locally constant if and only if $df = 0$.*

Proof. Use Remark 6.2. □

We finish this section by briefly discussing vector fields.

Definition. *Let $U \subseteq M$ be an open subset; a vector field ξ on U is a locally analytic map $\xi : U \longrightarrow T(M)$ which satisfies $p_M \circ \xi = \mathrm{id}_U$.*

It is easily seen that

$$\Gamma(U, T(M)) := \text{ set of all vector fields on } U \ .$$

is a K-vector space w.r.t. pointwise addition and scalar multiplication of maps.

Lemma 9.14. *For any vector field $\xi \in \Gamma(M, T(M))$ the map*

$$\begin{aligned} D_\xi : C^{\mathrm{an}}(M, K) &\longrightarrow C^{\mathrm{an}}(M, K) \\ f &\longmapsto df \circ \xi \end{aligned}$$

is a derivation, i. e.:

(a) D_ξ *is* K*-linear,*

(b) $D_\xi(fg) = D_\xi(f)g + fD_\xi(g)$ *for any* $f, g \in C^{\mathrm{an}}(M, K)$.

Proposition 9.15. *Suppose that* M *is paracompact; then for any derivation* D *on* $C^{\mathrm{an}}(M, K)$ *there is a unique vector field* ξ *on* M *such that* $D = D_\xi$.

Lemma 9.16. *For any derivations* $B, C, D : C^{\mathrm{an}}(M, K) \longrightarrow C^{\mathrm{an}}(M, K)$ *we have:*

i. $[B, C] := B \circ C - C \circ B$ *again is a derivation;*

ii. $[\ ,\ ]$ *is* K*-bilinear;*

iii. $[B, B] = 0$ *and* $[B, C] = -[C, B]$*;*

iv. (Jacobi identity) $[[B, C], D] + [[C, D], B] + [[D, B], C] = 0$.

Proof. These are straightforward completely formal computations. □

This lemma says that the vector space of derivations on $C^{\mathrm{an}}(M, K)$ is a K-Lie algebra. Using Prop. 9.15 it follows that $\Gamma(M, T(M))$ naturally is a Lie algebra (at least for paracompact M).

10 Reminder: Locally convex K-vector spaces

We recall very briefly the notion of a locally convex K-vector space. For details we refer to [NFA]. Let E be any K-vector space.

Definition. *A (nonarchimedean) seminorm on* E *is a function* $q : E \longrightarrow \mathbb{R}$ *such that for any* $v, w \in E$ *and any* $a \in K$ *we have:*

(i) $q(av) = |a| \cdot q(v)$,

(ii) $q(v + w) \leq \max(q(v), q(w))$.

Let $(q_i)_{i \in I}$ be a family of seminorms on E. We consider the coarsest topology on E such that:

(1) All maps $q_i : E \longrightarrow \mathbb{R}$, for $i \in I$, are continuous,

(2) all translation maps $v + . : E \longrightarrow E$, for $v \in E$, are continuous.

It is called the *topology defined by* $(q_i)_{i \in I}$.

Lemma 10.1. *E is a topological K-vector space, i. e., addition and scalar multiplication are continuous, with respect to the topology defined by $(q_i)_{i\in I}$.*

Exercise. *The topology on E defined by $(q_i)_{i\in I}$ is Hausdorff if and only if for any vector $0 \neq v \in E$ there is an index $i \in I$ such that $q_i(v) \neq 0$.*

Definition. *A topology on a K-vector space E is called locally convex if it can be defined by a family of seminorms. A locally convex K-vector space is a K-vector space equipped with a locally convex topology.*

Obviously any normed K-vector space and in particular any K-Banach space is locally convex.

Remark 10.2. *Let $\{E_j\}_{j\in J}$ be a family of locally convex K-vector spaces; then the product topology on $E := \prod_{j\in J} E_j$ is locally convex.*

For our purposes the following construction is of particular relevance. Let E be a any K-vector space, and suppose that there is given a family $\{E_j\}_{j\in J}$ of vector subspaces $E_j \subseteq E$ each of which is equipped with a locally convex topology.

Lemma 10.3. *There is a unique finest locally convex topology $\mathcal{T}$ on E such that all the inclusion maps $E_j \xrightarrow{\subseteq} E$, for $j \in J$, are continuous.*

The topology $\mathcal{T}$ on E in the above lemma is called the *locally convex final topology* with respect to the family $\{E_j\}_{j\in J}$. Suppose that the family $\{E_j\}_{j\in J}$ has the additional properties:

- $E = \bigcup_{j\in J} E_j$;
- the set J is partially ordered by $\leq$ such that for any two $j_1, j_2 \in J$ there is a $j \in J$ such that $j_1 \leq j$ and $j_2 \leq j$;
- whenever $j_1 \leq j_2$ we have $E_{j_1} \subseteq E_{j_2}$ and the inclusion map $E_{j_1} \xrightarrow{\subseteq} E_{j_2}$ is continuous.

In this case the locally convex K-vector space $(E, \mathcal{T})$ is called the *locally convex inductive limit* of the family $\{E_j\}_{j\in J}$.

Lemma 10.4. *A K-linear map $f : E \longrightarrow \tilde{E}$ into any locally convex K-vector space $\tilde{E}$ is continuous (with respect to $\mathcal{T}$) if and only if the restrictions $f|E_j$, for any $j \in J$, are continuous.*

11 The topological vector space $C^{\mathbf{an}}(M, E)$

Throughout this section M is a paracompact manifold and E is a K-Banach space. Following [Fea] we briefly describe how to equip $C^{\mathrm{an}}(M, E)$ with a locally convex topology.

Using that, by Prop. 8.7, M is strictly paracompact and that, by Lemma 1.4, open coverings of open subset in K^m can be refined into disjoint coverings by balls we obtain the following

Fact: *Given $f \in C^{\mathrm{an}}(M, E)$ there is a family of charts $(U_i, \varphi_i, K^{m_i})$, for $i \in I$, for M together with real numbers $\varepsilon_i > 0$ such that:*

(a) $M = \bigcup_{i \in I} U_i$, and the U_i are pairwise disjoint;

(b) $\varphi_i(U_i) = B_{\varepsilon_i}(x_i)$ for one (or any) $x_i \in \varphi_i(U_i)$;

(c) there is a power series $F_i \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; E)$ with

$$f \circ \varphi_i^{-1}(x) = F_i(x - x_i) \qquad \text{for any } x \in \varphi_i(U_i) \ .$$

Let (c, ε) be a pair consisting of a chart $c = (U, \varphi, K^m)$ for M and a real number $\varepsilon > 0$ such that $\varphi(U) = B_\varepsilon(a)$ for one (or any) $a \in \varphi(U)$. As a consequence of the identity theorem for power series Cor. 5.8 the K-linear map

$$\begin{aligned} \mathcal{F}_\varepsilon(K^m; E) &\longrightarrow C^{\mathrm{an}}(U, E) \\ F &\longmapsto F(\varphi(.) - a) \end{aligned}$$

is injective. Let $\mathcal{F}_{(c,\varepsilon)}(E)$ denote its image. It is a K-Banach space with respect to the norm

$$\|f\| = \|F\|_\varepsilon \quad \text{if} \quad f(.) = F(\varphi(.) - a) \ .$$

By Cor. 5.5 the pair $(\mathcal{F}_{(c,\varepsilon)}(E), \| \ \|)$ is independent of the choice of the point a.

Definition. *An index for M is a family $\mathcal{I} = \{(c_i, \varepsilon_i)\}_{i \in I}$ of charts $c_i = (U_i, \varphi_i, K^{m_i})$ for M and real numbers $\varepsilon_i > 0$ such that the above conditions (a) and (b) are satisfied.*

For any index $\mathcal{I}$ for M we have

$$\mathcal{F}_{\mathcal{I}}(E) := \prod_{i \in I} \mathcal{F}_{(c_i, \varepsilon_i)}(E) \subseteq \prod_{i \in I} C^{\mathrm{an}}(U_i, E) = C^{\mathrm{an}}(M, E) \ .$$

Our above Fact says that

$$C^{\mathrm{an}}(M,E) = \bigcup_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(E)$$

where $\mathcal{I}$ runs over all indices for M. Hence $C^{\mathrm{an}}(M,E)$ is a union of direct products of Banach spaces, which are locally convex by Remark 10.2. Therefore, by Lemma 10.3, we may equip $C^{\mathrm{an}}(M,E)$ with the corresponding locally convex final topology. All our earlier constructions involving $C^{\mathrm{an}}(M,E)$ are compatible with this topology. In the following we briefly list the most important ones.

Proposition 11.1. *For any $a \in M$ the evaluation map*

$$\begin{aligned} \delta_a : C^{\mathrm{an}}(M,E) &\longrightarrow E \\ f &\longmapsto f(a) \end{aligned}$$

is continuous.

Corollary 11.2. *The locally convex vector space $C^{\mathrm{an}}(M,E)$ is Hausdorff.*

Remark 11.3. *With M also its tangent bundle $T(M)$ is paracompact.*

Proposition 11.4. *i. The map $d : C^{\mathrm{an}}(M,E) \longrightarrow C^{\mathrm{an}}(T(M),E)$ is continuous.*

ii. For any locally analytic map of paracompact manifolds $g : M \longrightarrow N$ the map

$$\begin{aligned} C^{\mathrm{an}}(N,E) &\longrightarrow C^{\mathrm{an}}(M,E) \\ f &\longmapsto f \circ g \end{aligned}$$

is continuous.

iii. For any vector field ξ on M the map $D_\xi : C^{\mathrm{an}}(M,E) \longrightarrow C^{\mathrm{an}}(M,E)$ is continuous.

Proposition 11.5. *For any covering $M = \bigcup_{i\in I} U_i$ by pairwise disjoint open subsets U_i we have*

$$C^{\mathrm{an}}(M,E) = \prod_{i\in I} C^{\mathrm{an}}(U_i,E)$$

as topological vector spaces.

To prove these properties one needs Lemma 10.4. This requires to see that $C^{\mathrm{an}}(M,E)$, in fact, is the locally convex inductive limit of the $\mathcal{F}_{\mathcal{I}}(E)$. Technically this means that one has to introduce a directed preorder $\mathcal{I} \leq \mathcal{J}$ (on the set of indices) such that $\mathcal{F}_{\mathcal{I}}(E) \subseteq \mathcal{F}_{\mathcal{J}}(E)$ with this inclusion being continuous.

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