Ehrhart polynomials of matroids and hypersimplices

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Definition of Matroid

Let us recall briefly what a *matroid* is:

Definition

Let E be a finite set and let $\mathscr{B} \subseteq 2^E$ be a family of subsets of E satisfying:

1. $\mathscr{B} \neq \varnothing$.
2. If A and B are distinct members of $\mathscr{B}$ and $a \in A \smallsetminus B$, then there exists $b \in B \smallsetminus A$ such that $(A \smallsetminus \{a\}) \cup \{b\} \in \mathscr{B}$.

We say that $M = (E, \mathscr{B})$ is a *matroid* on E, and call the elements $B \in \mathscr{B}$ the *bases* of M.

Rank and independence

Remark

- One can prove that all the sets $B \in \mathscr{B}$ must have the same cardinality, which we may denote with the integer k. We say that k is the *rank* of M.
- We say that a set $I \subseteq E$ is *independent* if it is contained in some $B \in \mathscr{B}$.

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Remark

The family $\mathscr{I}$ of all independent sets of a matroid M is a pure simplicial complex. It satisfies a nice property: all its induced subcomplexes are pure. This characterizes all *matroid complexes*.

From matroid to polytopes

Definition

Let M be a matroid on the set $E = \{1, \ldots, n\}$ with set of bases $\mathscr{B}$ and let $\mathscr{I}$ all the independent subsets of M. For each $A \subseteq E$ let us define a point in $\mathbb{R}^n$ by $e_A = \sum_{i \in A} e_i$, where e_i is the i-th canonical vector.

- We define the *matroid polytope* or *basis polytope* of M as:

$$\mathscr{P}(M) \doteq \textit{convex hull}\ \{e_B : B \in \mathscr{B}\} \subseteq \mathbb{R}^n$$

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- We define the *independence (matroid) polytope* of M as:

$$\mathscr{P}_I(M) \doteq \textit{convex hull}\ \{e_I : I \in \mathscr{I}\} \subseteq \mathbb{R}^n$$

From polytopes to matroids?

Theorem (GGMS '87)

A polytope $\mathscr{P}$ is the basis polytope of a matroid if and only if:

(a) *All the vertices of $\mathscr{P}$ have $0/1$ coordinates.*

(b) *All the edges of $\mathscr{P}$ are of the form $e_i - e_j$.*

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Theorem (Edmonds '69)

Let $\mathscr{P} \subseteq \mathbb{R}^n$ be the independence polytope of a matroid. Then:

- *All the vertices of $\mathscr{P}$ have $0/1$ coordinates.*
- *All the edges of $\mathscr{P}$ are of the form $e_i - e_j$, e_i or $-e_i$.*

Generalized Permutohedra

Definition (Postnikov '09)

A *generalized permutohedron* is a polytope $\mathscr{P} \subseteq \mathbb{R}^n$ that has each of its edges is parallel to some $e_i - e_j$ for $i \neq j$.

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Thus, basis polytopes are exactly GP with 0/1 coordinates.
What about independence polytopes?

Proposition (Ardila et al. '11 - LF '20)

There is a "canonical" way to lift the independence matroid polytope $\mathscr{P}_I(M)$ and obtain a generalized permutohedron $\widetilde{\mathscr{P}_I}(M)$.

Our lifting is an integral equivalence. This means that it preserves, for example, the *Ehrhart polynomial* of our independence matroid polytope.

The Ehrhart Polynomial

Theorem (Ehrhart '62)

If $\mathscr{P} \subseteq \mathbb{R}^n$ is a lattice polytope of dimension m, then the function

$$i(\mathscr{P}, t) = \#(t\mathscr{P} \cap \mathbb{Z}^n),$$

defined for $t \in \mathbb{N}$ is a polynomial of degree m.

An example

Example

Let $U_{2,4}$ be the uniform matroid with 4 elements and rank 2. It is defined on the ground set $E = \{1, 2, 3, 4\}$ and its set of bases is:

$$\mathscr{B} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

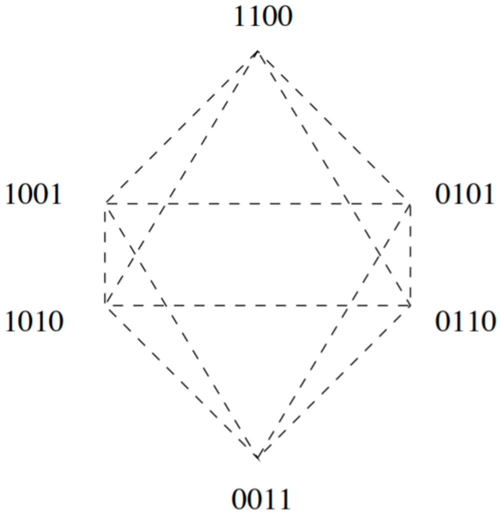
The matroid polytope is given by the convex hull of the following points in $\mathbb{R}^4$:

$$\{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$$

This defines a polytope of dimension 3 in $\mathbb{R}^4$, also known as the *hypersimplex* $\Delta_{2,4}$. The Ehrhart polynomial is:

$$i(U_{2,4},t) = \frac{2}{3}t^3 + 2t^2 + \frac{7}{3}t + 1.$$

An example



Ehrhart positivity

Conjecture (De Loera, Haws, Köppe '07)

The Ehrhart polynomial of a matroid polytope always has positive coefficients.

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This is not true for general polytopes. In fact there exist counterexamples for 0/1-polytopes.

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In 2015 a *stronger* conjecture was formulated by Castillo and Liu:

Conjecture (Castillo, Liu '15)

All generalized permutohedra with vertices with integer coordinates are Ehrhart positive.

The case of uniform matroids

Theorem (LF '19)

The basis polytopes of all uniform matroids are Ehrhart positive.

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Theorem (LF '20)

The independence matroid polytopes of all uniform matroids are Ehrhart positive.

The latter is a consequence of a nicer fact: all *half-open hypersimplices* are Ehrhart Positive. The proof of both results is combinatorial, and do not allow us to get much geometric insight.

An upper bound

It is easy to show that if M is a matroid of rank k and cardinality n, then its matroid polytope $\mathscr{P}$ is *contained* in the hypersimplex $\Delta_{n,k}$.
Therefore one has the inequality

$$i(M, t) \leq i(U_{k,n}, t)$$

for all $t \in \mathbb{Z}_{\geq 0}$. We conjecture something stronger:

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Conjecture (LF '20)

If M is a matroid of rank k and cardinality n then, for all $0 \leq m \leq n-1$, the m-th coefficient of $i(M, t)$ is less or equal than the m-th coefficient of $i(U_{k,n}, t)$.

A lower bound?

This motivates us to look for another matroid whose Ehrhart-coefficients could be a possible *lower bound*. Without loss of generality we may restrict ourselves only to *connected matroids* (those that are not a direct sum of smaller matroids).

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Proposition (Dinolt, Murty)

For all n and k, the least number of bases a connected matroid of rank k and cardinality n can have is exactly $k(n-k)+1$. There is only one matroid up to isomorphisms for which this minimum is attained.

Minimal matroids

We will denote this unique matroid of size n and rank k by $T_{k,n}$.
It is given by the cycle matroid of a graph given by a cycle of length $k+1$ where one of the edges is replaced by $n-k$ parallel copies.

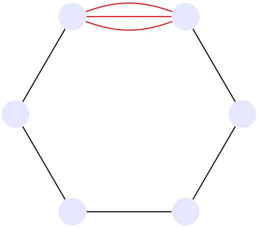

Figure: $T_{5,8}$

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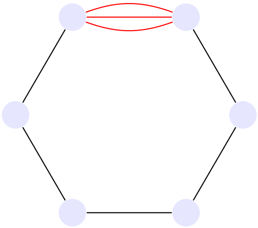

Figure: $T_{5,8}$

Remark

It can be proven that $T_{k,n}$ is isomorphic to the *snake matroid* $S(k, n-k)$ (Knauer - Martínez - Ramírez).

The Ehrhart polynomial of $T_{k,n}$

Theorem (LF '20)

$$i(T_{k,n}, t) = \frac{1}{\binom{n-1}{k-1}} \binom{t+n-k}{n-k} \sum_{j=0}^{k-1} \binom{n-k-1+j}{j} \binom{t+j}{j}$$

In particular $i(T_{k,n}, t-1)$ has positive coefficients (and hence $T_{k,n}$ is Ehrhart-positive).

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In particular $i(T_{k,n}, t-1)$ has positive coefficients (and hence $T_{k,n}$ is Ehrhart-positive).

Conjecture

If M is a connected matroid of rank k and size n, then for all $1 \leq m \leq n-1$, the m-th coefficient of $i(M,t)$ is greater or equal than the m-th coefficient of $i(T_{k,n}, t)$.

A proof of this conjecture would imply De Loera et al's conjecture.

h^*-polynomials

Theorem (Stanley '93)

Let $\mathscr{P}$ be a lattice polytope of dimension m. Then:

$$\sum_{k=0}^{\infty} i(\mathscr{P}, k)x^k = \frac{h^*(x)}{(1-x)^{m+1}},$$

for a polynomial h^* of degree at most m and nonnegative integer coefficients.

This suggests that the coefficients of the h^*-polynomial of an integral polytope are counting something. For uniform matroids we have such interpretations.

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Theorem (Li 12', Early 17', Kim 20')

There is a combinatorial interpretation of the coefficients of the h^*-polynomial of all hypersimplices $\Delta_{k,n}$.

h^*-polynomials of matroids

Conjecture (De Loera et al '07)

The h^*-polynomial of a matroid polytope has unimodal coefficients. This means that if we write $h^*(x) = h_0 + h_1 x + \ldots + h_m x^m$, there is an index $0 \leq j \leq m$ such that:

$$h_0 \leq h_1 \leq \cdots \leq h_{j-1} \leq h_j \geq h_{j+1} \geq \cdots \geq h_m.$$

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The only infinite families of matroids for which this Conjecture has been proved are minimal matroids (Knauer et al 18' or Ferroni 20') and snake matroids of the form $S(a, \ldots, a)$ (Knauer et al 18').

Real-rootedness

Proposition

If a polynomial p has positive coefficients and all of its roots are real numbers, then p has log-concave coefficients and, in particular, they are unimodal.

Conjecture (Ferroni 20')

The h^*-polynomial of matroid polytopes are real-rooted.

Particular cases are still open. It is an open problem to prove that h^*-polynomials of hypersimplices are real-rooted.

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Question (Real rootedness for $h^*(\Delta_{2,n}, x)$)

$$h^*(x) = 1 + \left(\binom{n}{2} - n\right)x + \binom{n}{4}x^2 + \binom{n}{6}x^3 + \binom{n}{8}x^4 + \ldots + \binom{n}{2\lfloor\frac{n}{2}\rfloor}x^{\lfloor\frac{n}{2}\rfloor}.$$

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3. All uniform matroids with up to 200 elements.

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4. All matroids with up to 9 elements.

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3. All uniform matroids with up to 200 elements.
4. All matroids with up to 9 elements.
5. All lattice-path-matroids with up to 12 elements.

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6. All snake matroids $S(a_1, \ldots, a_k)$ with $a_1 + \ldots + a_k \leq 22$.

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7. All matroids listed in [De Loera - Haws - Köppe '07].

THANK YOU