Flow / transportation polytope volume bounds via polynomial capacity

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Outline

1. Polynomial capacity
   - Used to bound/approximate coefficients of log-concave polynomials
2. Log-concave polynomials
   - Notions of log-concavity, examples, and capacity bounds
3. Contingency tables (CT)
   - Generating function is log-concave
   - Approximately count CTs using capacity
4. Flow / transportation polytopes
   - CTs are the integral points of these polytopes
   - Limit CT capacity bounds to obtain volume bounds

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Polynomial capacity

Let $\mathbb{R}_+[\boldsymbol{x}]$ denote the set of n-variate polynomials with all coefficients ≥ 0.

Definition: Given $p \in \mathbb{R}_+[\boldsymbol{x}]$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{x_1,x_2,\ldots,x_n>0} \frac{p(x_1,x_2,\ldots,x_n)}{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}.$$

Intuitions/interpretations of capacity:

1. **Combinatorial:** $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) > 0$ iff $\boldsymbol{\alpha} \in \mathrm{Newt}(p) = \mathrm{hull}(\mathrm{supp}(p))$.
2. **Entropic:** $\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p)$ is the entropy of a special distribution on $\mathrm{supp}(p)$ with expectation $\boldsymbol{\alpha}$.
3. **Convexity/optimization:** $\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p)$ can be converted into a convex program, and can be approximated if p is easy to evaluate.

Key use: Approximation of polynomial coefficients:

$$\mathrm{Cap}_{\boldsymbol{\kappa}}(p) \geq \langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq K(\boldsymbol{\kappa}) \cdot \mathrm{Cap}_{\boldsymbol{\kappa}}(p).$$

Key idea: We use evaluations of p to approximate coefficients of p.

Gurvits' original application: Computing permanents

Given a matrix M with entries in $\mathbb{R}_+$, define the **permanent** of M:

$$\mathrm{per}(M) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

Barvinok (I think?): "Like the determinant, but simpler." **Hilarious!**

Why? Exact permanent computation is the canonical #P-hard problem.

Already #P-hard for 0-1 matrices, which is equivalent to counting perfect matchings of a bipartite graph.

Relation to capacity? Defining $q(\boldsymbol{x}) := \prod_{i=1}^n \sum_{j=1}^n m_{ij} x_j$, we have

$$\mathrm{per}(M) = \langle x_1 x_2 \cdots x_n \rangle q(\boldsymbol{x}) = \partial_{x_1} \partial_{x_2} \cdots \partial_{x_n} q(\boldsymbol{x}).$$

Upshot: q is easy to evaluate, but the coefficients are hard to compute.

Coefficient approximation

Last slide: Given M with $\mathbb{R}_+$ entries, we define $q(\boldsymbol{x}) = \prod_{i=1}^n \sum_{j=1}^n m_{ij} x_j$ which implies $\mathrm{per}(M) = \langle \boldsymbol{x}^{\mathbf{1}} \rangle q(\boldsymbol{x})$.

Theorem (Gurvits '08)

Given any d-homogeneous, n-variate, "strongly log-concave" (SLC) $p \in \mathbb{R}_+[\boldsymbol{x}]$ *and any* $\boldsymbol{\kappa} \in \mathrm{supp}(p)$, *we have*

$$\mathrm{Cap}_{\boldsymbol{\kappa}}(p) \geq \langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq \binom{d}{\boldsymbol{\kappa}} \frac{\kappa_1^{\kappa_1} \cdots \kappa_n^{\kappa_n}}{d^d} \mathrm{Cap}_{\boldsymbol{\kappa}}(p).$$

Upshot: Coefficient approximation via capacity (a convex program).

Apply to q with $\boldsymbol{\kappa} = \mathbf{1}$: $\mathrm{Cap}_{\mathbf{1}}(q) \geq \mathrm{per}(M) \geq \frac{n!}{n^n} \cdot \mathrm{Cap}_{\mathbf{1}}(q)$.

Next: Coefficients of generating functions count combinatorial objects.

Corollary: If "SLC" generating function, we can approximately count.

What is "SLC"? What other classes of log-concave polynomials?

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Log-concave polynomials

SLC / Completely log-concave / Lorentzian (homogeneous):

- **E.g.:** $\det(\sum_i x_i A_i)$ for PSD A_i, $\mathrm{vol}(\sum_i x_i K_i)$ for compact convex K_i, matroid basis generating polynomials [ALOV '18, BH '19], products of linear forms with non-negative entries, more?
- **Hot:** Hodge theory for matroids [AHK '15], Mason's conjectures [ALOV '18, BH '19]
- $\sum_{k=0}^{d} c_k x^k y^{d-k}$ is SLC $\iff \left(\frac{c_k}{\binom{d}{k}}\right)^2 \geq \left(\frac{c_{k-1}}{\binom{d}{k-1}}\right)\left(\frac{c_{k+1}}{\binom{d}{k+1}}\right)$.

Denormalized Lorentzian (homogeneous): Multiplying the coefficients of p by multinomial coefficients gives a Lorentzian polynomial.

- **E.g.:** Schur polynomials [HMMD '19], polymatroid basis generating polynomials, contingency tables generating polynomials, more?
- $\sum_{k=0}^{d} c_k x^k y^{d-k}$ is DL $\iff c_k^2 \geq c_{k-1} \cdot c_{k+1}$.

Bonus: Both classes preserved under taking products of polynomials.

Capacity and denormalized Lorentzian polynomials

Before: Coefficient bounds for SLC polynomials via capacity.

Theorem (Brändén-L-Pak '20)

Given any d-homogeneous, n-variate, denormalized Lorentzian (DL) $p \in \mathbb{R}_+[\boldsymbol{x}]$ and any $\boldsymbol{\kappa} \in \mathrm{supp}(p)$, we have

$$\mathrm{Cap}_{\boldsymbol{\kappa}}(p) \geq \langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq e^{-(n-1)} \left[\prod_{i=2}^{n} \frac{1}{\kappa_i + 1} \right] \mathrm{Cap}_{\boldsymbol{\kappa}}(p).$$

Now: We can approximate the coefficients of DL polynomials.

- Coefficients of Schur polynomials. **Applications?**
- Counting contingency tables. **Next section.**

(**Quick comment:** Starting at $i = 2$ is not a typo.)

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Contingency tables

Definition: Given $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$, a **contingency table (CT)** is an $m \times n$ matrix of non-negative integers such that the row sums and column sums are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ called the **marginals** of M).

Examples: Contingency tables with $\boldsymbol{\alpha} = (1, 4)$ and $\boldsymbol{\beta} = (1, 2, 2)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

The permutation matrices are the contingency tables with $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$.

Generating function: Fix matrix M, and to entry m_{ij} associate $(x_i y_j)^{m_{ij}}$. M has marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ iff $\prod_{i=1}^m \prod_{j=1}^n (x_i y_j)^{m_{ij}} = \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}$. **Therefore:**

$$g(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \prod_{i=1}^m \prod_{j=1}^n \sum_{k=0}^\infty (x_i y_j)^k = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}},$$

where $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ counts contingency tables with the given marginals.

Capacity bounds for contingency tables

Goal: Apply capacity bounds to generating function.

Problems: Not a polynomial, not homogeneous. **We can fix it though:**

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\sum_{k=0}^{\infty}(x_i y_j)^k \;\to\; \prod_{i=1}^{m}\prod_{j=1}^{n}\sum_{k=0}^{K} x_i^k y_j^{K-k} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{mK\cdot\mathbf{1}-\boldsymbol{\beta}},$$

where $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the number of tables with entries bounded by K.

Upshot: New generating function is a product of bivariate homogeneous polynomials $\sum_{k=0}^{K} x_i^k y_j^{K-k}$ with log-concave coefficients.

Therefore: The new generating function is denormalized Lorentzian.

Finally: Apply capacity bound, then un-twist and send $K \to \infty$ to get:

$$\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{\beta_j + 1}\right] \mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(g).$$

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Transportation/flow polytopes

Definition: For $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, the **transportation polytope** $\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the set of all $m \times n$ matrices with $\mathbb{R}_+$ entries and marginals $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. (**Flow polytopes** have extra constraint that entries are bounded by k_{ij}.)

That is: Contingency tables are the integer points of these polytopes.

Idea: We can extract volume from $\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})$ as $M \to \infty$.

How? If h_P is the Ehrhart polynomial of an integral polytope P, then $h_P(M)$ counts integer points in $M \cdot P$ for $M \in \mathbb{N}$. **So:**

$$h_{\mathcal{T}(\boldsymbol{\alpha},\boldsymbol{\beta})}(M) = h_{\mathcal{T}(M\boldsymbol{\alpha},M\boldsymbol{\beta})}(1) = \mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta}).$$

Well known: The leading coefficient of h_P is $\mathrm{vol}(P)$.

Therefore: Since the dimension of $\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is $(m-1)(n-1)$, we have

$$\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})) = \lim_{M \to \infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}.$$

Next: We can bound $\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta}))$ by limiting our capacity bound on CTs.

Volume bounds via capacity

Last slide: $\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha},\boldsymbol{\beta})) = \lim_{M\to\infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}$.

From before, we have our capacity bound:

$$\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta}) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{M\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{M\beta_j + 1}\right] \mathrm{Cap}_{(M\boldsymbol{\alpha},M\boldsymbol{\beta})}(g)$$

Now add in the limit:

$$\lim_{M\to\infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{mn-(m+n-1)}} = e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j}\right] \lim_{M\to\infty} \frac{\mathrm{Cap}_{(M\boldsymbol{\alpha},M\boldsymbol{\beta})}(g)}{M^{mn}}$$

Last piece:

$$\lim_{M\to\infty} \frac{\mathrm{Cap}_{(M\boldsymbol{\alpha},M\boldsymbol{\beta})}(g)}{M^{mn}} = \inf_{0<\boldsymbol{x},\boldsymbol{y}<1} \frac{\prod_{i=1}^{m}\prod_{j=1}^{n} \frac{-1}{\log(x_i y_j)}}{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}}.$$

This can also be converted into a convex program.

Volume bounds via capacity

Theorem (Brändén-L-Pak '20)

Given $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$, the volume of $\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be bounded via

$$\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j} \right] \cdot \inf_{0<\boldsymbol{x},\boldsymbol{y}<1} \frac{\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{-1}{\log(x_i y_j)}}{\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}}.$$

The same holds for the flow polytope with $\frac{-1}{\log(x_i y_j)}$ replaced by $\frac{(x_i y_j)^{k_{ij}} - 1}{\log(x_i y_j)}$.

Corollary: For $\boldsymbol{\alpha} = \alpha_0 \cdot \mathbf{1}$ and $\boldsymbol{\beta} = \beta_0 \cdot \mathbf{1}$, we obtain a **closed-form** bound.

For the **Birkhoff polytope** with $\boldsymbol{\alpha} = \boldsymbol{\beta} = 1$ and $m = n$:

$$\mathrm{vol}(\mathcal{T}_{\mathbf{1},\mathbf{1}}) \geq \frac{(en)^{(n-1)^2}}{n^{2n^2-2n+1}} = \frac{e^{(n-1)^2}}{n^{n^2}} = e^{-n^2 \log n + n^2 - 2n + 1}.$$

First two terms coincide with the true asymptotics [Canfield-McKay '07].

Questions?

(And thanks for listening!)