Monotone Paths on Cross-Polytopes

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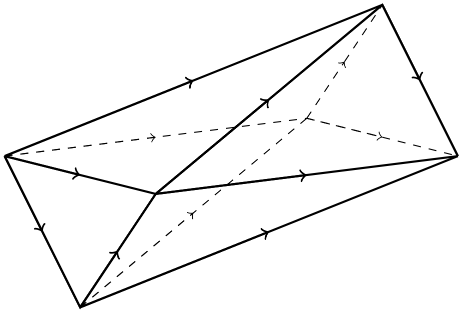
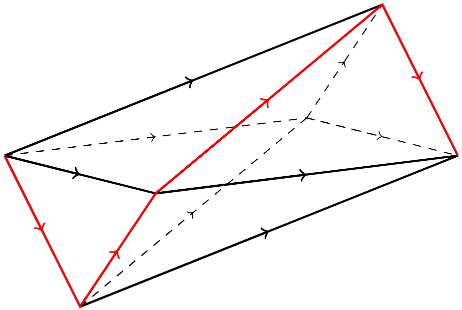
Generic Orientation

Any edge generic linear functional induces a unique sink orientation on the graph of a polytope:



Monotone Paths

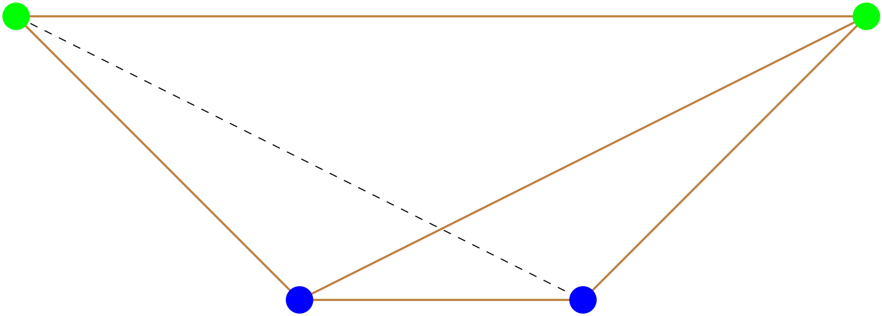
A monotone path is a path from the minimum to the maximum on the oriented one-skeleton of a polytope with orientation induced by a generic linear functional:



Monotone Paths on Simplices

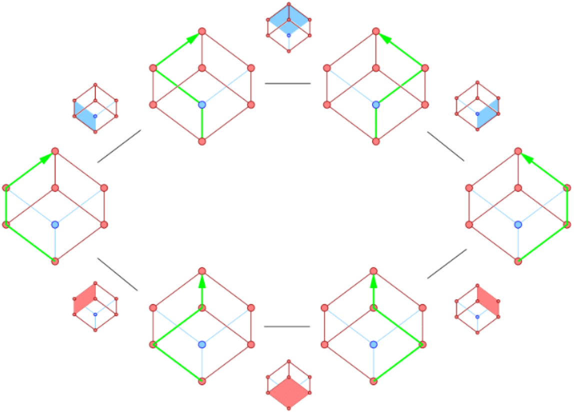
Exercise

The set of all monotone paths on the simplex Δ^n (or more generally any neighborly polytope with $n+1$ vertices) is in bijection with subsets of $[n-1]$.



Flip Graph

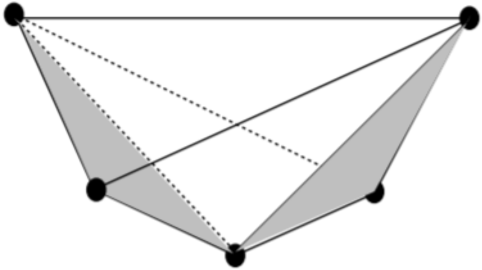
One may form a graph of monotone paths called the flip graph, where two monotone paths are adjacent if they agree everywhere except on a two-dimensional face.



Cellular Strings

Definition

A cellular string is a sequence of increasing faces of a polytope connected end to end.



Baues Poset

Definition

The Baues poset is the poset of all cellular strings under refinement.

Theorem (Billera and Sturmfels, 1992)

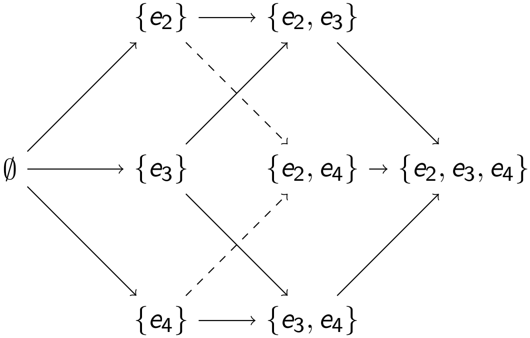
The order complex of the Baues poset has a canonical deformation retraction onto a polytope. We call this polytope the monotone path polytope $MPP_{\varphi}(P)$. *Its faces correspond to coherent cellular strings with its vertices being coherent monotone paths.*

Monotone Path Polytopes of Simplices

Every cellular string on a simplex is coherent!

Theorem (Billera and Sturmfels, 1992)

A monotone path polytope of a simplex Δ^n vertices for a generic linear functional is combinatorially equivalent to a hyper-cube C_{n-1}.

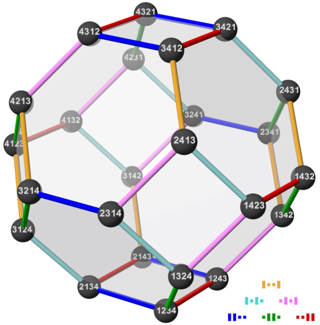


Monotone Path Polytopes of Cubes

Every cellular string on the cube is coherent!

Theorem (Billera and Sturmfels, 1992)

A monotone path polytope of a cube C_n for a generic linear functional is normally equivalent to the permutahedron $conv(\{S_n(1, 2, \ldots, n)\})$.

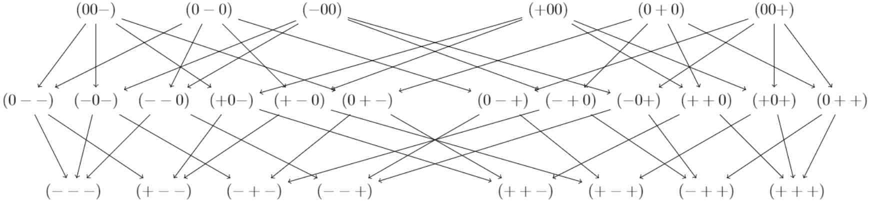


What about cross-polytopes?

Not every monotone path on the cross-polytope is coherent.

Theorem (B. and De Loera, 2021)

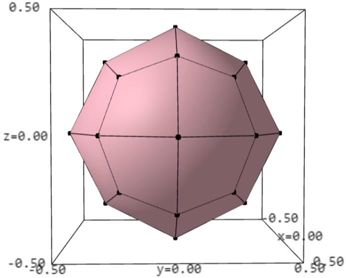
The face lattice of the monotone path polytope of a cross-polytope $\diamond^{n-1}$ *for a generic linear functional is equivalent to the lattice of intervals in the sign poset:* $\{+, -, 0\}^{n-1} \setminus \{\vec{0}\}$.

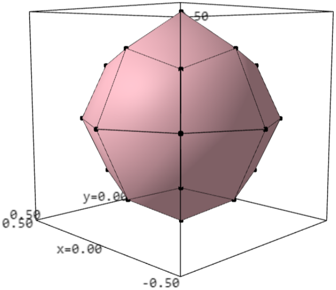


Combinatorially Equivalent Form: Signohedron

Theorem (B. and De Loera, 2021)

For any generical linear functional φ, $MPP_{\varphi}(\diamond^n)$ is combinatorially equivalent to $(\diamond^{n-1} + C_{n-1})^*$.

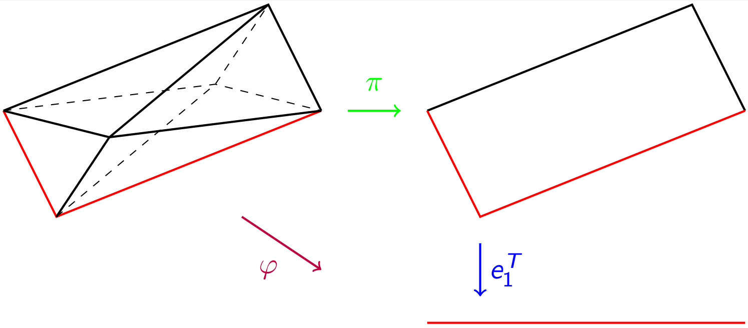




But what does coherence mean?

Definition

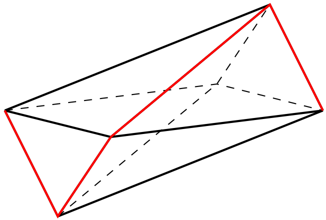
A monotone path on a polytope P is coherent if there exists a two-dimensional projection of P taking the path to the lower edges of a polygon.

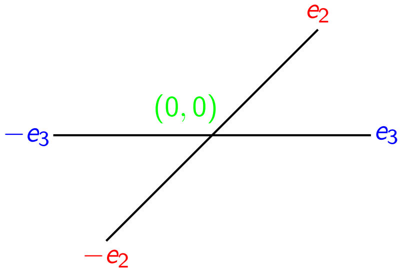


Incoherence on Centrally Symmetric Polytopes

Corollary (B. and De Loera, 2021)

A monotone path on a centrally symmetric polytope containing a pair of antipodes other than the max and min cannot be coherent.

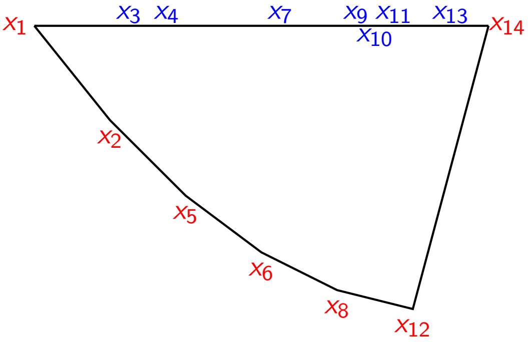




Coherence on the Simplex

Theorem (Billera and Sturmfels, 1992)

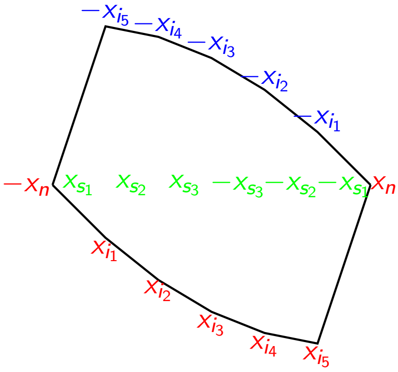
All cellular strings on the simplex are coherent.



Coherence on the Cross-Polytope

Theorem (B. and De Loera)

A cellular string on the cross-polytope is coherent if and only if it does not contain a pair of antipodes other than the max and min.



The Sign Poset Correspondence

Theorem (B. and De Loera, 2021)

Coherent monotone paths on the cross-polytope are in bijection with elements of the sign poset.

(i) If e_i is in the path $x_i = +$

(ii) If $-e_i$ is in the path $x_i = -$

(iii) If neither are in the path $x_i = 0$

For example,

$$-e_4, -e_2, e_3, e_4 \mapsto (0, -, +)$$

The Sign Poset Correspondence

Theorem (B. and De Loera, 2021)

Coherent cellular strings on the cross-polytope are in bijection with intervals in the sign poset.

A coherent cellular string is determined by two things:

(i) Endpoints of the string

(ii) The vertices in each cell

Each of (i) and (ii) corresponds to the vertices of a monotone path and thus a sign vector. The cellular string corresponds to the interval between those sign vectors.

Why the cross-polytope?

Lemma (Billera and Sturmfels, 1992)

Let P, Q be polytopes, and $\theta : P \to Q$ be a surjective linear map and $\varphi : Q \to \mathbb{R}$ be a linear functional. Then

$$\theta(MPP_{\varphi \circ \theta}(P)) = MPP_{\varphi}(Q)$$

Projections take monotone path polytopes to monotone path polytopes.

Corollary (B. and De Loera, 2021)

The monotone path polytopes of centrally symmetric polytopes are projections of combinatorial signohedra.

Description of the Signohedra

Proposition (B. and De Loera, 2021)

The $f-$vector of the signohedron $MPP_\varphi(\diamond^n)$ is given by

$$f_m(MPP_\varphi(\diamond^n)) = \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^{k+m}$$
$$= 2^m \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^k.$$

Proposition (B. and De Loera, 2021)

$$diam(MPP_\varphi(\diamond^n)) = 2(n-1) = (n-1)diam(\diamond^n).$$

Explicit Polyhedral Realization

Proposition (B. and De Loera, 2021)

$$MPP_{\varphi}(\diamond^n) = \{x \in \mathbb{R}^n : \pi(x) = 0 \textit{ and } \varphi_{i,\varepsilon}(x) \geq -a_i - a_n$$
$$\textit{for all } \varepsilon : [n-1] \to \{\pm 1\}, k \in [n-1]\}$$

where we define $\varphi_{i,\varepsilon}$ *on the basis* $F_1 \cup F_2 \cup \{e_n\}$ *by*

$$\varphi_{i,\varepsilon}(e_k) = \begin{cases} -a_k - a_n \textit{ if } k \in F_1 \\ \frac{a_i + a_n}{a_n - a_i}(a_k - a_n) \textit{ if } k \in F_2 \\ 0 \textit{ if } k = n \end{cases}$$

for $F_1 = \{k : \varepsilon(k)k \leq i\}$ *and* $F_2 = \{k : \varepsilon(k)k \geq i\}$.

Explicit Vertex Realization

Proposition (B. and De Loera, 2021)

The set of vertices of the Signohedron is given by

$$\left\{ \left(1 - \frac{a_{i_k} + a_{i_1}}{2a_n}\right) e_n + \sum_{i=1}^{k} \left(\frac{a_{i_{k-1}} + a_{i_{k+1}}}{2a_n}\right) e_{i_k} : \right.$$

$$\left. -n = i_0 < \cdots < i_{k+1} = n \text{ and } i_a \neq -i_b \text{ for all } a, b \in [k] \right\}.$$

The Plot of $MPP_\varphi(\diamond^4)$