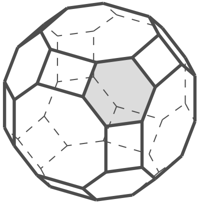
# Generators for type B permutahedra via McMullen's polytope algebra



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Cornell University

(Polytop)ics: Recent advances on polytopes
Max Planck Institute for Mathematics in the Sciences
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# Question

(Posed by Ardila-Castillo-Eur-Postnikov '20)

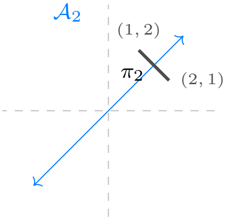
What is a *nice* type B analog of the following result?

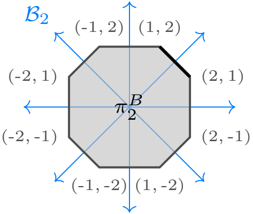
**Theorem (Postnikov '09, Ardila-Benedetti-Doker '10)**

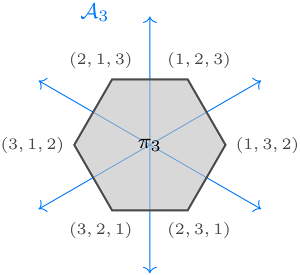
*Every generalized permutahedron in $\mathbb{R}^d$ can be written* ***uniquely*** *as a signed Minkowski sum of the faces of the standard simplex $\Delta_{[d]}$.*

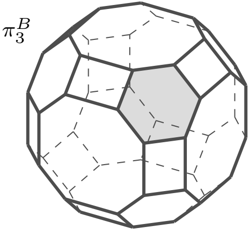
# The (type B) permutahedron

$$\pi_d = \mathrm{Conv}\{\sigma \cdot (1, 2, \ldots, d) \, : \, \sigma \in \mathfrak{S}_d\} \qquad \pi_d^B = \mathrm{Conv}\{\tau \cdot (1, 2, \ldots, d) \, : \, \tau \in \mathfrak{S}_d^{\pm}\}$$



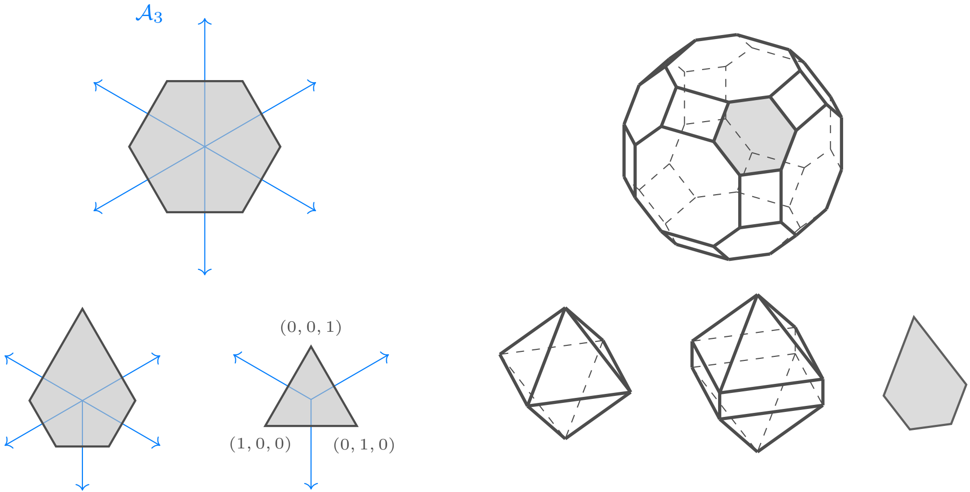



# Generalized permutahedra

Generalized (type B) permutahedra are deformations of $\pi_d^{(B)}$.

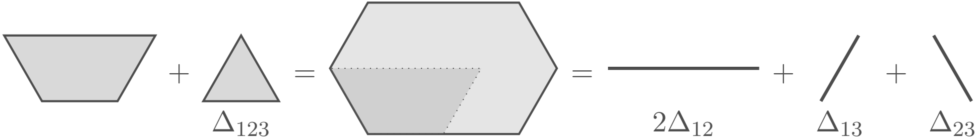



Their normal fan is refined by the corresponding *Coxeter fan*.

# Signed Minkowski sum

The **Minkowski sum** of two polytopes $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{R}^d$ is the polytope

$$\mathfrak{p} + \mathfrak{q} = \{a + b \,:\, a \in \mathfrak{p},\, b \in \mathfrak{q}\}.$$




We can express the trapezoid above as the signed **Minkowski sum**:

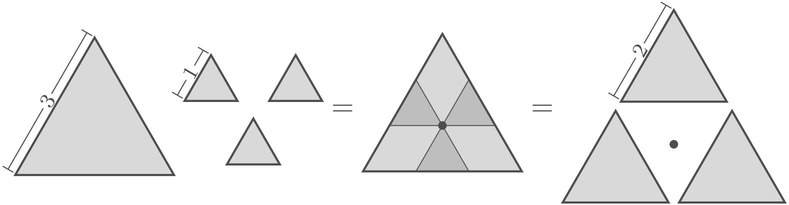

$$= 2\Delta_{12} + \Delta_{13} + \Delta_{23} - \Delta_{123}$$

# McMullen's polytope algebra

Let $\Pi(\mathbb{R}^d)$ be the abelian group generated by *classes* of polytopes $[\mathfrak{p}]$, with relations:

- $[\mathfrak{p} \cup \mathfrak{q}] + [\mathfrak{p} \cap \mathfrak{q}] = [\mathfrak{p}] + [\mathfrak{q}]$, whenever $\mathfrak{p} \cup \mathfrak{q}$ is a polytope.
- $[\mathfrak{p} + \{t\}] = [\mathfrak{p}]$ for any *translation* $t \in \mathbb{R}^d$.

**Example:** Let $\triangle$ be a 2-simplex. Then, $[3\triangle] + 3[\triangle] = 3[2\triangle] + [\cdot]$



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**Example:** Let $\triangle$ be a 2-simplex. Then, $[3\triangle] + 3[\triangle] = 3[2\triangle] + [\cdot]$.

- The product in $\Pi(\mathbb{R}^d)$ is given by Minkowski sum:

$$[\mathfrak{p}] \cdot [\mathfrak{q}] = [\mathfrak{p} + \mathfrak{q}] \qquad \begin{pmatrix} 1 = [\cdot] \\ [\mathfrak{p}]^n = [n\mathfrak{p}] \end{pmatrix}$$

The previous example can be restated as $([\triangle] - 1)^3 = 0$.

# McMullen's polytope algebra

In fact, for any polytope $\mathfrak{p} \subseteq \mathbb{R}^d$, $([\mathfrak{p}] - 1)^{\dim(\mathfrak{p})+1} = 0$. So we can define

$$\log[\mathfrak{p}] := \sum_{k=1}^{\dim(\mathfrak{p})} \frac{(-1)^{k-1}}{k}([\mathfrak{p}] - 1)^k$$

**Theorem (McMullen '89)**

$\Pi(\mathbb{R}^d)$ *is a graded ring:*

$$\Pi(\mathbb{R}^d) = \bigoplus_{r=0}^{d} \Pi_r(\mathbb{R}^d).$$

*The space* $\Pi_1(\mathbb{R}^d)$ *is spanned by elements of the form* $\log[\mathfrak{p}]$.

$$\log[\mathfrak{p} + \mathfrak{q}] = \log([\mathfrak{p}] \cdot [\mathfrak{q}]) = \log[\mathfrak{p}] + \log[\mathfrak{q}]$$
$$\log[n\mathfrak{p}] = \log([\mathfrak{p}]^n) = n \log[\mathfrak{p}]$$

So signed Minkowski sums become linear combinations in $\Pi_1(\mathbb{R}^d)$.

# Subalgebra of generalized permutahedra

Let $\Pi(\pi_d^B) = \bigoplus_{r=0}^{d} \Pi_r(\pi_d^B)$ be the subalgebra of $\Pi(\mathbb{R}^d)$ generated by classes of type B generalized permutahedra.

Coming up:

- $\Pi(\pi_d^B)$ is a module over the Tits algebra of the type B arrangement.
- The module structure of $\Pi_1(\pi_d^B)$ will give us information about *generating* collections of type B generalized permutahedra.

# Tits monoid of a hyperplane arrangement

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$.

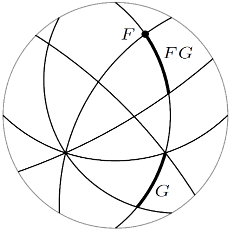
- $\mathcal{L} = \mathcal{L}[\mathcal{A}]$ denotes the lattice of *flats* of $\mathcal{A}$.

  flat = intersection of some hyperplanes of $\mathcal{A}$

- $\Sigma = \Sigma[\mathcal{A}]$ denotes the collection of *faces* of $\mathcal{A}$.

  face = face of a closed region of $\mathcal{A}$

$\Sigma$ is a monoid: The product of two faces $F$ and $G$, denoted $FG$, is the first face you encounter after moving a small positive distance from an interior point of $F$ to an interior point of $G$. The central face $O$ is the unit of this product.



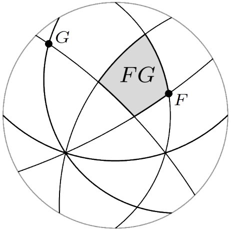




Image borrowed from:
M. Aguiar, S. Mahajan, *Topics in hyperplane arrangements*, Mathematical Surveys and Monographs, vol. 226, AMS, 2017.

# Polytope algebra as a module

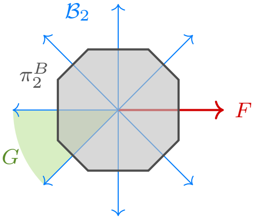
The Tits algebra $\mathbb{R}\Sigma$ is the monoid algebra of $\Sigma$.

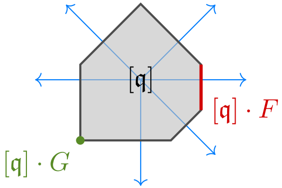
**Theorem (B- '20)**

*Let $\mathfrak{p}$ be a zonotope of $\mathcal{A}$, then $\Pi(\mathfrak{p})$ is a right $\mathbb{R}\Sigma$-module:*

$$[\mathfrak{q}] \cdot F = [\textit{face of } \mathfrak{q} \textit{ maximized in the direction of } F]$$

*and the action of each $F \in \Sigma$ is an algebra endomorphism.*




- Each graded component $\Pi_r(\mathfrak{p})$ is a $\mathbb{R}\Sigma$-submodule.

# Invariants

Simple modules over $\mathbb{R}\Sigma$ are well understood (Solomon '67, Bidigare '97):

- They are one-dimensional.
- There is one isomorphism class $M_{\mathrm{X}}$ for each flat $\mathrm{X} \in \mathcal{L}$.

But $\mathbb{R}\Sigma$ is not semisimple.

What simple modules appear in a composition series of $\Pi_r(\pi_d^B)$?

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = \Pi_r(\pi_d^B)$$

(The *composition factors* $M_{i+1}/M_i$ are simple $\mathbb{R}\Sigma$-modules)

$\eta_{\mathrm{X}}(\Pi_r(\pi_d^B))$ denotes the number of composition factors isomorphic to $M_{\mathrm{X}}$.

# Main result

Let $\mathfrak{S}_d^{\pm}$ be the Coxeter group of type $B_d$ (signed permutations).

**Theorem (B- '20)**

*For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{B}_d]$ *and* $r = 0, 1, \ldots, d$,

$$\eta_{\mathrm{X}}(\Pi_r(\pi_d^B)) = \#\{\tau \in \mathfrak{S}_d^{\pm} \,:\, \mathrm{fix}(\tau) = \mathrm{X},\ \mathrm{exc}(\tau) + \lfloor \frac{\mathrm{neg}(\tau) + 1}{2} \rfloor = r\}.$$

Tools: McMullen 83', Brenti '94, Aguiar-Mahajan '17.

# The module $\Pi_1(\pi_d^B)$

**Corollary (B- '20)**

*Any family of generators (via signed Minkowski sums) for generalized permutahedra of type B contains at least $2^{d-1}$ full-dimensional polytopes.*

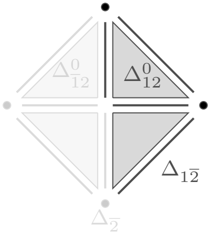
14 full-dimensional *type B shard polytopes* in $\mathbb{R}^3$ (Padrol-Pilaud-Ritter '20)

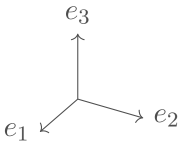
- From the previous theorem, $\eta_{\{\mathbf{0}\}}(\Pi_1(\pi_d^B)) = 2^{d-1}$.
- We employ Eulerian idempotents of $\mathbb{R}\Sigma[\mathcal{B}_d]$ (Saliola '06, Aguiar-Mahajan '17) to deduce that, if $\mathfrak{q}$ is not full-dimensional, the projection of $\log[\mathfrak{q}]$ to certain $\eta_{\{\mathbf{0}\}}(\Pi_1(\pi_d^B))$-dimensional subspace is zero.

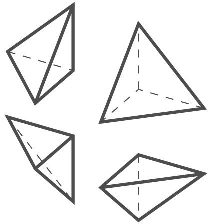
# Generators for type B generalized permutahedra

For nonempty $S \subseteq [\pm d]$ with $S \cap \overline{S} = \emptyset$ define

$$\Delta_S = \mathrm{Conv}\{e_j \mid j \in S\} \quad \text{and} \quad \Delta^0_S = \mathrm{Conv}(\Delta_S \cup \{\mathbf{0}\}) \qquad (e_{\bar{i}} = -e_i)$$



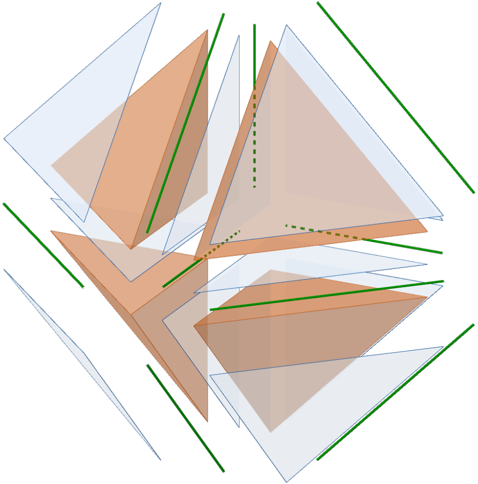
**Theorem (B-)**

*Any type B generalized permutahedron can be written uniquely as a signed Minkowski sum of the simplices*

$$\{\Delta_S, \Delta^0_S \;:\; \min\{|j| \;:\; j \in S\} \in S\}$$

The proof uses a *valuation* $\Pi_1(\pi^B_d) \to \mathbb{R}\Sigma[\mathcal{B}_d]$.

# Thank you!



arXiv:2009.05876