A q-analogue of Brion's theorem from quantum K-theory

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Overview

1. Brion's identity recap

2. q-analogue of Brion's identity

Lattice point generating functions

Definition

For $K \subset \mathbb{R}^n$, let

$$\sigma_K(x) = \sum_{u \in K \cap \mathbb{Z}^n} x^u$$

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For us, K will be either

- a lattice polytope P, in which case $\sigma_P(x)$ is a Laurent polynomial, or
- a rational polyhedral cone C, in which case $\sigma_C(x)$ is a rational function, see e.g. the textbook [BR].

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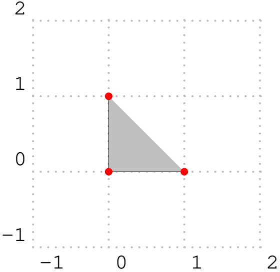

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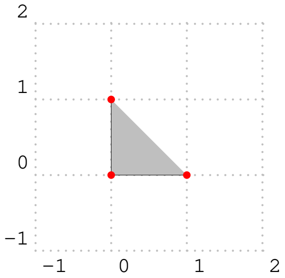

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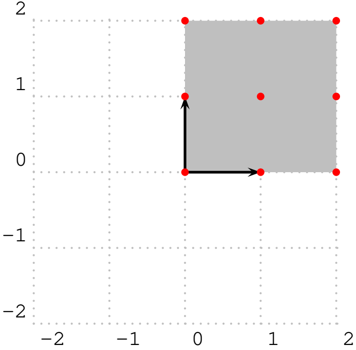

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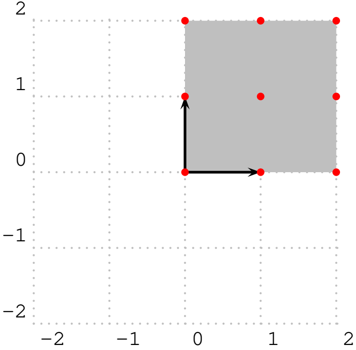

then $\sigma_C(x) = 1 + x + y + x^2 + xy + y^2 + \ldots = \frac{1}{(1-x)(1-y)}$.

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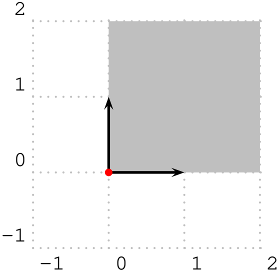

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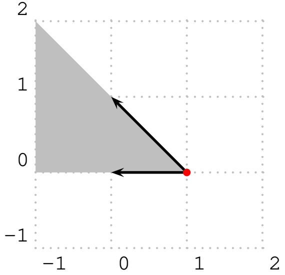

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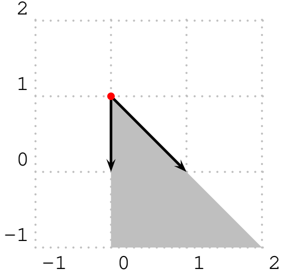

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Something to notice is that summing these functions up recovers $\sigma_P(x)$:

$$\sigma_{K_{(0,0)}P}(x) + \sigma_{K_{(1,0)}P}(x) + \sigma_{K_{(0,1)}P}(x) = \sigma_P(x).$$

Brion's identity

Theorem (Brion, 1988)

This happens for P any simple lattice polytope.

Note: This theorem has been reproved and generalized. However, it was first proved using the Atiyah-Bott-Berline-Vergne-(Baum-Fulton-Quart-...) integration formula in equivariant K-theory, on the toric variety associated to P.

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Definition

Let

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$$

be the *q-Pochhhammer symbol*. Let

$$\binom{n}{k_1, k_2, \ldots, k_l}_q = \frac{(q;q)_n}{\prod_{i=1}^{l}(q;q)_{k_i}}$$

be the q-multinomial coefficient.

Note that these quantities all specialize to 1 if $q = 0$, and if $q \to 1$, then $\binom{n}{k_1,k_2,\ldots,k_l}_q \to \binom{n}{k_1,k_2,\ldots,k_l}$.

Some more notation

Suppose P has r facets, with primitive inward normal vectors $v_1, \ldots, v_r$. Then we can realize P as the intersection of half-spaces

$$P = \bigcap_{i=1}^{r} \{u | \langle u, v_i \rangle \geq -a_i\},$$

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where a_i is a nonnegative integer. We assume P is *smooth*, i.e. if for each vertex p, $K_p P$ is a generated by a basis for $\mathbb{Z}^n$. Also, we assume P is *radially symmetric*, meaning $\sum_{i=1}^{r} v_i = 0$.

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Definition

Suppose P is radially symmetric. Then, define the generalized Rogers-Szegő polynomial

$$RS_P(x, q) = \sum_{u \in P \cap \mathbb{Z}^n} \binom{\sum_i a_i}{\langle u, v_1 \rangle + a_1, \ldots, \langle u, v_r \rangle + a_r}_q x^u.$$

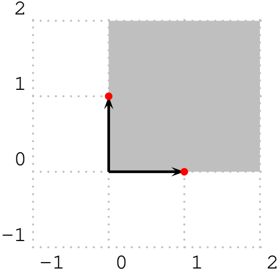
Example for the interval

Let $P = [0, k]$, then $RS_P(x, q)$ is the k-th classical Rogers-Szegő polynomial [S].

$$RS_P(x, q) = RS_k(x, q) = \sum_{i=0}^{k} \binom{k}{i}_q x^i.$$

Just a little more notation

Let p be a vertex of P. Then, let $u_1(p), \ldots, u_n(p)$ be the primitive vectors along the edges of K_pP which are incident to p. In the example of $(0, 0)$ in the simplex, we just have the highlighted vectors:



Main Theorem

[Anderson-S, 2021]

For P a smooth, radially symmetric polytope,

$$RS_P(x,q) = \frac{(q;q)_{\sum_i a_i}}{(q;q)_\infty^{r-n}} \sum_{p \in V(P)} \frac{x^p}{(x^{u_1(p)};q)_\infty \dots (x^{u_n(p)};q)_\infty} \phi_{P,p}(x,q),$$

where $\phi_{P,p}$ is an explicit q-hypergeometric series.

Note: We will see an example of $\phi_{P,p}$. When the associated toric variety is *Fano*, this function is a specialization of the K-theoretic J-function as studied in Gromov-Witten theory. When $q = 0$, this becomes Brion's identity

$$\sigma_P(x) = \sum_{p \in V(P)} \frac{x^p}{(1 - x^{u_1(p)}) \dots (1 - x^{u_n(p)})}.$$

The case of intervals

Let P be the interval $[0, k]$. Then the left-hand side of our main theorem is the k-th Rogers-Szegő polynomial [S]:

$$RS_P(x, q) = RS_k(x, q) = \sum_{i=0}^{k} \binom{k}{i}_q x^i,$$

and

$$\phi_{P,0} = \sum_{i=0}^{\infty} \frac{q^{ki}}{(q^{-i}; q)_i (xq^{-i}; q)_i}$$

$$\phi_{P,k} = \sum_{i=0}^{\infty} \frac{q^{ki}}{(q^{-i}; q)_i (x^{-1}q^{-i}; q)_i}$$

so, the main theorem states

$$RS_k(x, q) = \frac{(q; q)_k}{(q; q)_\infty} \left(\frac{\phi_{P,0}}{(x; q)_\infty} + x^k \frac{\phi_{P,k}}{(x^{-1}; q)_\infty} \right)$$

Degenerate case:

The degenerate case of our main theorem where the polytope is a point also holds. For intervals, this is the identity

$$(q;q)_\infty = \frac{1}{(x;q)_\infty}\sum_{i=0}^{\infty}\frac{1}{(q^{-i};q)_i(xq^{-i};q)_i} + \frac{1}{(x^{-1};q)_\infty}\sum_{i=0}^{\infty}\frac{1}{(q^{-i};q)_i(x^{-1}q^{-i};q)_i}$$

which (with a little work) can be rearranged to the Jacobi Triple Product identity:

$$\prod_{i\geq 1}(1-q^i)(1-q^{i-1}x)(1-q^i/x) = \sum_{i\in\mathbb{Z}}(-1)^i x^i q^{i(i-1)/2}.$$

Bonus slide for the curious: the definition of $\phi_{P,p}$

Definition

The vectors v_i, which are inward normal to the facets, define a map $\mathbb{Z}^r \to \mathbb{Z}^n$. Let A be the kernel, and denote the inclusion of A into $\mathbb{Z}^r$ by β. Let A_+ be the intersection of A with $\mathbb{Z}^r_{\geq 0}$.
For p a vertex of P, let $I(p) \subset \{1, \ldots, r\}$ be the set of indices i so that v_i in the inward normal of a facet containing p. Then v_i is a basis for $\mathbb{Z}^n$, let $u_i(p)$ be the dual basis. Then

$$\phi_{P,p} = \sum_{d \in A_+} \frac{q^{a_i \beta(d)_i}}{\prod_{i \in I(p)} (x^{u_i(p)} q^{-1}; q^{-1})_{\beta(d)_i} \prod_{j \notin I(p)} (q^{-1}; q^{-1})_{\beta(d)_j}}$$

References

Beck, M. and Robins, S.
Computing the continuous discretely,
Springer, New York (2015)

Brion, M.
"Points entiers dans les polyèdres convexes,"
Ann. Sci. École Norm. Sup. (4) 21 (1988), 653–663

Szegő. G.
"Ein Beitrag zur Theorie der Thetafunktionen,"
Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse, (1926), 242–252

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