# Sweep polytopes and sweep oriented matroids

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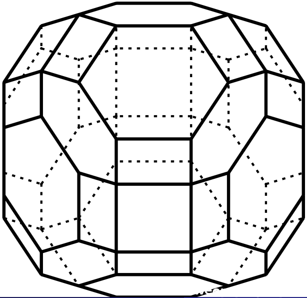
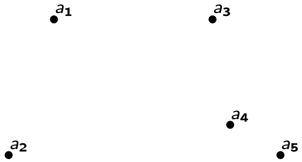


Joint work with Arnau Padrol

arXiv:2102.06134

# Sommaire

$A = \{a_1, \dots, a_n\}$ a configuration of $n$ points in $\mathbb{R}^d$.



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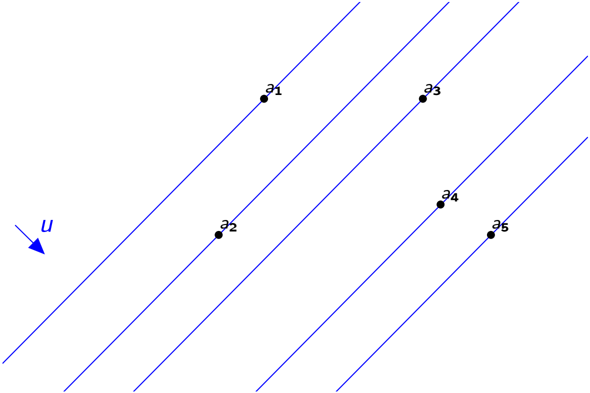


*Sweep permutation* 1, 2, 3, 4, 5.

$$\langle u\,,\, a_1\rangle < \langle u\,,\, a_2\rangle < \langle u\,,\, a_3\rangle < \langle u\,,\, a_4\rangle < \langle u\,,\, a_5\rangle\,.$$

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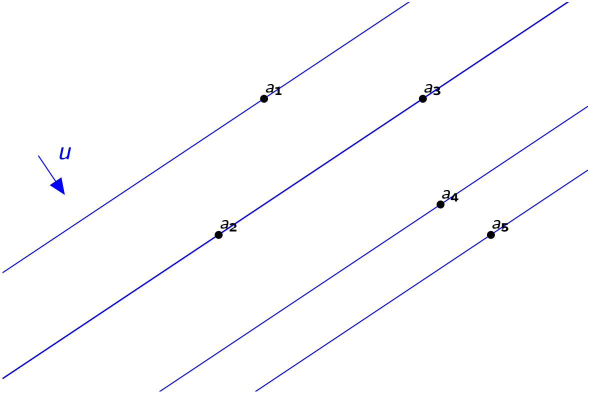
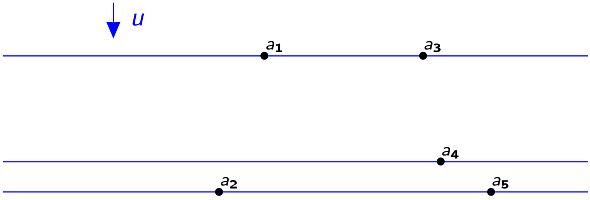



*Sweep* $1, 23, 4, 5$.

$$\langle u\,,\, a_1\rangle < \langle u\,,\, a_2\rangle = \langle u\,,\, a_3\rangle < \langle u\,,\, a_4\rangle < \langle u\,,\, a_5\rangle\,.$$

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54321

In dimension 2, Perrin (1882), Goodman and Pollack (1980-1993) studied *allowable sequences*.

Edelman (2000) and Stanley (2015) gave bounds on the number of sweep permutations (or *valid order arrangements* of an affine hyperplane arrangement).

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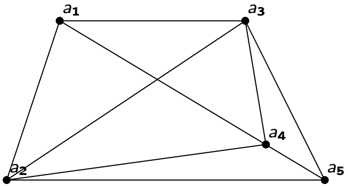
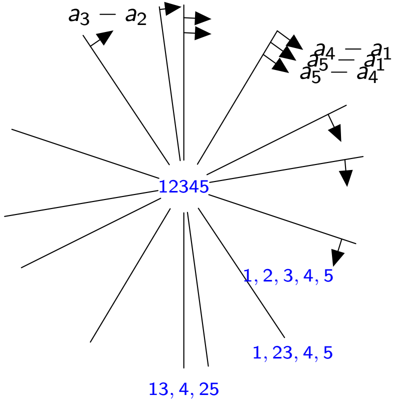


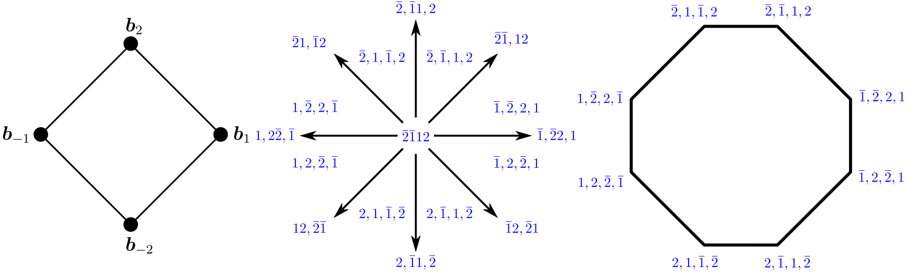

$\mathcal{SH}(A)$ = arrangement of the hyperplanes $\{u \in \mathbb{R}^d \mid \langle u, a_i \rangle = \langle u, a_j \rangle\}$, for all $(i,j) \in \binom{[n]}{2}$.
Its cones are in bijection with the sweeps of $A$.

# Sweep polytope

## Definition

The *sweep polytope* $\boldsymbol{SP}(A)$ of the point configuration $A$ is the *zonotope* :

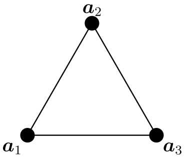
$$\boldsymbol{SP}(A) = \sum_{1 \leq i < j \leq n} \left[ -\frac{a_i - a_j}{2}, \frac{a_i - a_j}{2} \right].$$

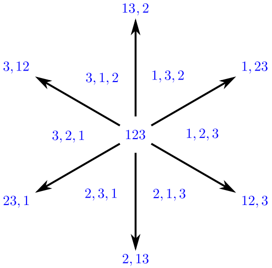


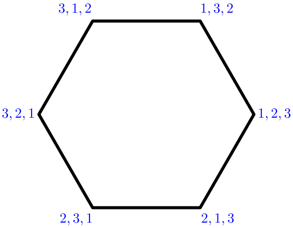
# Sweeps of the simplex : the permutahedron

$A = \{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$.

$$\boldsymbol{P}_n = \sum_{1 \le i < j \le n} \left[ -\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2} \right] = \text{conv}\left( \left\{ \sum_{i=1}^{n} \sigma(i) e_i \, , \, \sigma \in \mathfrak{S}_n \right\} \right) - \frac{n+1}{2} \sum_{i=1}^{n} e_i .$$



# Projection of the permutahedron

$A = \{a_1, \ldots, a_n\}$ a configuration of $n$ points in $\mathbb{R}^d$.
We consider the projection

$$\begin{aligned} M_A : \mathbb{R}^n &\rightarrow \mathbb{R}^d \\ e_i &\mapsto a_i. \end{aligned}$$

Then $\boldsymbol{SP}(A) = M_A(\boldsymbol{P}_n)$.

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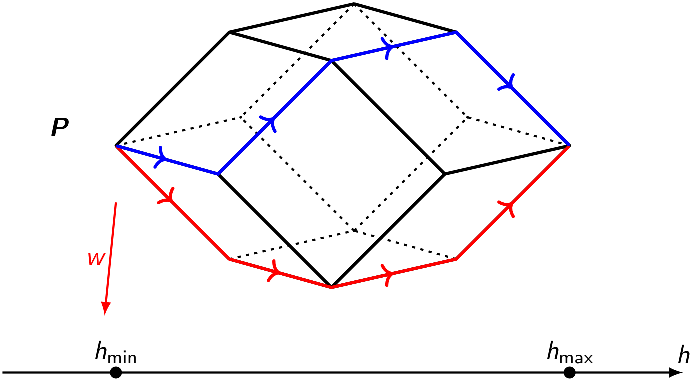
Then $\boldsymbol{SP}(A) = M_A(\boldsymbol{P}_n)$.

Conversely, if $M : \mathbb{R}^n \to \mathbb{R}^d$ is a linear map, $M(\boldsymbol{P}_n)$ is a sweep polytope.

# Monotone path polytope

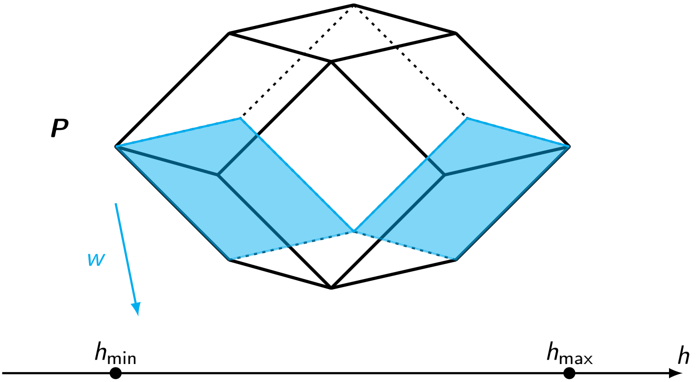



$$\Sigma(\boldsymbol{P}, h) = \left\{ \frac{1}{h_{\max} - h_{\min}} \int_{h_{\min}}^{h_{\max}} \gamma(y)\, dy \,\middle|\, \gamma \text{ is a section of } h \right\}.$$

**Theorem (Billera-Sturmfels, 1992)**

*The vertices of $\Sigma(\boldsymbol{P}, h)$ are in bijection with the coherent $h$-monotone paths of $\boldsymbol{P}$.*

# Monotone path polytopes

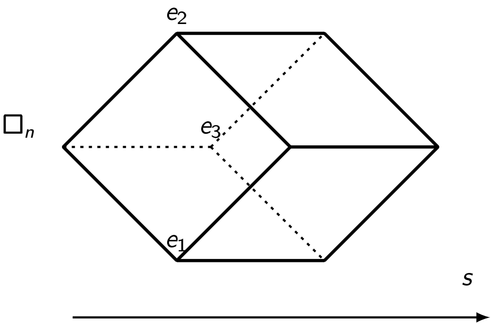



**Theorem (Billera-Sturmfels, 1992)**

*The face poset of $\Sigma(\boldsymbol{P}, h)$ is isomorphic to the poset of coherent h-cellular strings of $\boldsymbol{P}$.*

# The permutahedron, monotone path polytope of the cube

$\square_n = [-1,1]^n$, $s : x \in \mathbb{R}^n \mapsto \sum_{i=1}^n x_i$.




$$\Sigma(\square_n, s) = \frac{2}{n} \boldsymbol{P}_n.$$

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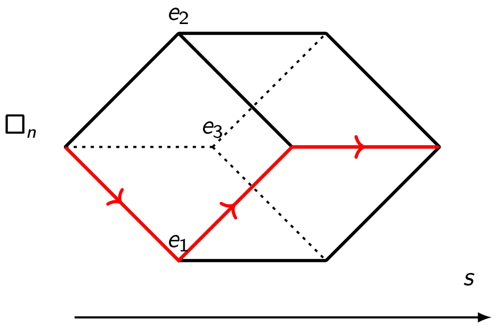
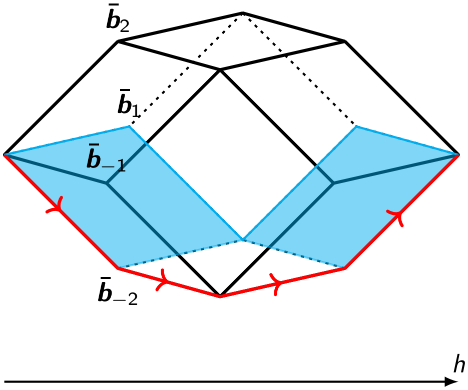
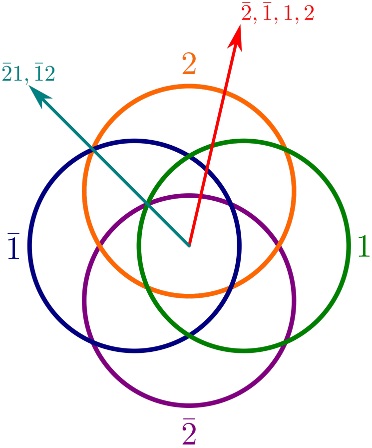



$$\Sigma(\square_n, s) = \frac{2}{n}\boldsymbol{P}_n.$$

# The sweep polytope as a monotone path polytope

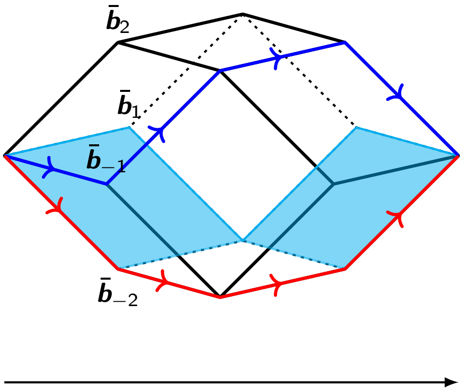
$\boldsymbol{Z}_{\bar{A}} = \sum_{i=1}^{n} [-(a_i, 1), (a_i, 1)] \subset \mathbb{R}^{d+1}$, $h : x \in \mathbb{R}^{d+1} \mapsto x_{d+1}$

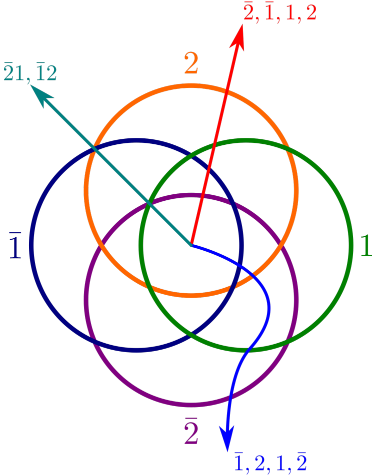



$$\Sigma(\tfrac{n}{2} \boldsymbol{Z}_{\bar{A}}, h) = \boldsymbol{SP}(A) \times \{0\}.$$

# Non-coherent paths and pseudo-sweeps



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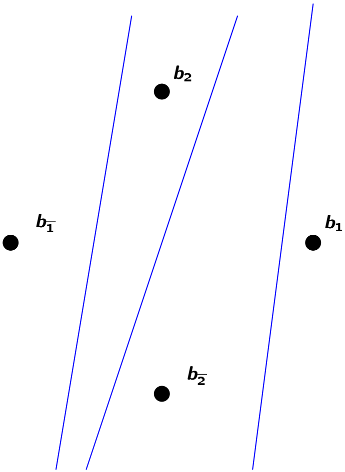
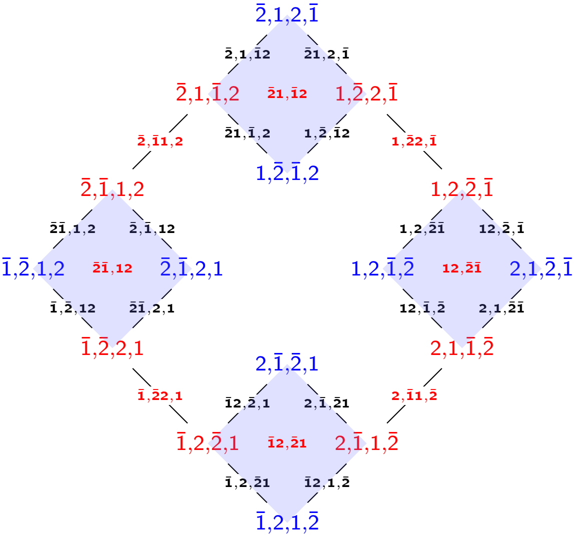
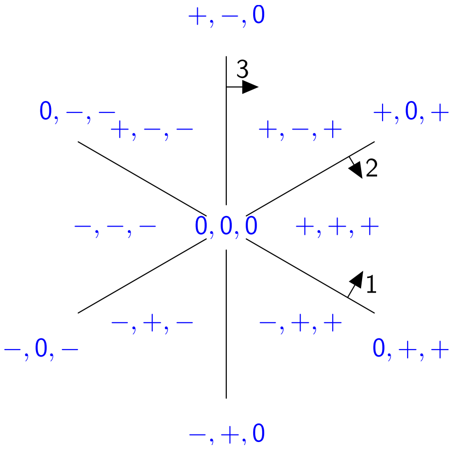
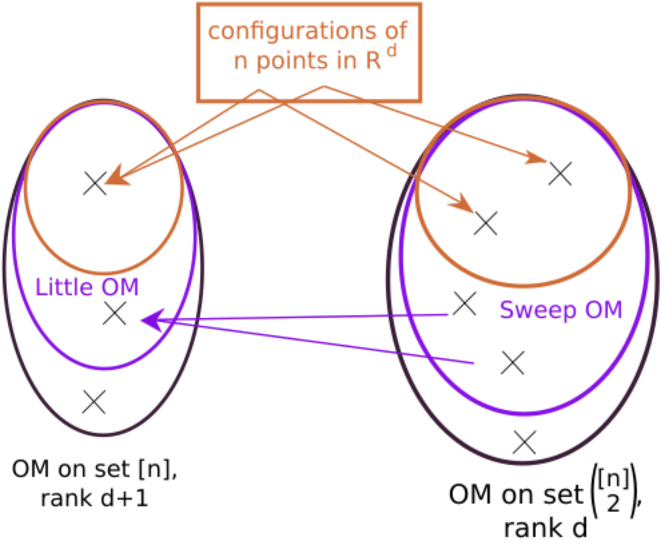


# Sweeps and pseudo-sweeps



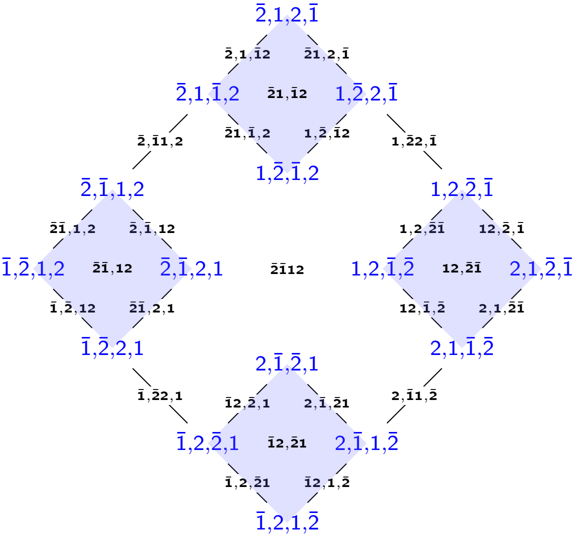
An *oriented matroid* $\mathcal{M}$ on ground set $E$ can be described as a set of elements in $\{+,-,0\}^E$ that satisfies certain properties.






*Sweep oriented matroids* are oriented matroids whose covectors can be associated to ordered partitions.

# Generalized Baues problem



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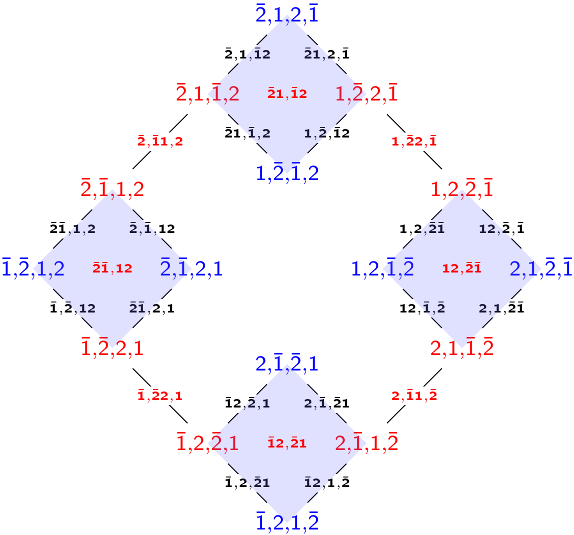
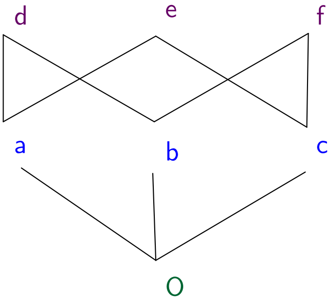
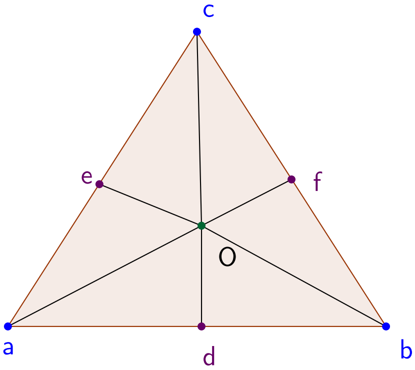


# Topology of posets

The *order complex* of a poset $\mathcal{P}$ is the simplicial complex of its chains.



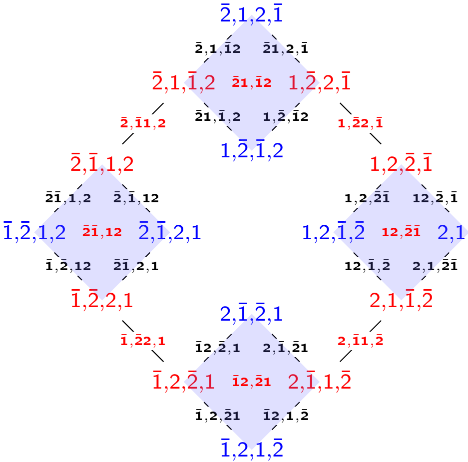



# Generalized Baues Problem for cellular strings of a zonotope

$\boldsymbol{P}$ a $d$-polytope, $h$ a linear functional in $\mathbb{R}^d$

$\omega(\boldsymbol{P}, h) :=$ non trivial cellular strings,
$\omega_{coh}(\boldsymbol{P}, h) :=$ non trivial coherent cellular strings.

**Weak GBP.** Is $\omega(\boldsymbol{P}, h)$ homotopy equivalent to a $(d-2)$-sphere ?
**Strong GBP.** Is $\omega_{coh}(\boldsymbol{P}, h)$ a deformation retract of $\omega(\boldsymbol{P}, h)$ ?



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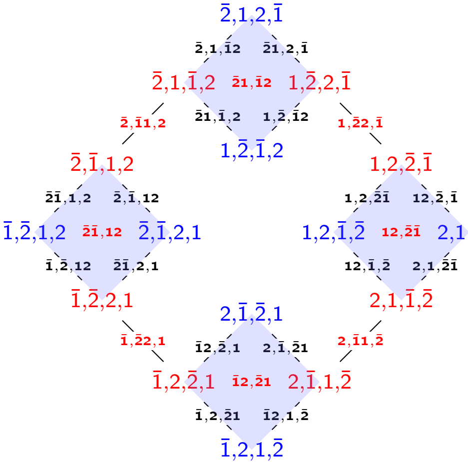
Answer : YES
(Billera-Kapranov-Sturmfels, 1992)



# Generalized Baues Problem for cellular strings of oriented matroids

$\mathcal{M}$ an oriented matroid of rank $r$, $T$ one of its tope.

**Weak GBP.** Is $\omega(\mathcal{M}, T)$ homotopy equivalent to a $(d-2)$-sphere ?

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Answer : YES (Björner, 1992)

**Strong GBP.** Does $\omega(\mathcal{M}, T)$ retract to a subcomplex homeomorphic to a $(d-2)$-sphere ?

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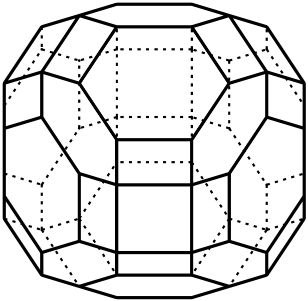
## Theorem (Padrol-P., 2021)

If $\mathcal{M}$ is the little oriented matroid of a sweep oriented matroid $\mathcal{M}^{\text{sw}}$. Then the poset of non-trivial sweeps of $\mathcal{M}^{\text{sw}}$ is a $(r-2)$-sphere and a strong deformation retract of $\omega(\mathcal{M}, +)$.



Thank you for your attention !