Castelnuovo polytopes

Akiyoshi Tsuchiya (University of Tokyo)

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Polarized toric varieties and lattice polytopes

X : an n-dimensional complex projective variety
L : an ample line bundle on X

We call the pair (X, L) an n-dimensional polarized variety.

$\mathcal{P} \subset \mathbb{R}^n$: a full-dimensional lattice polytope
(i.e., the convex hull of finitely many lattice points in $\mathbb{Z}^n$)

$$\{(X, L) : \text{a polarized toric variety}\}$$

$$\overset{1:1}{\longleftrightarrow}$$

$$\{\mathcal{P} : \text{a full-dimensional lattice polytope}\}$$

We can translate many terms of polarized toric varieties (algebraic geometry) into terms of lattice polytopes (combinatorics).

Sectional genus

(X, L) : an n-dimensional polarized variety
$\chi(tL)$: the Euler–Poincaré characteristic of tL

Then $\chi(tL)$ is a polynomial in t and we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \binom{t+j-1}{j}.$$

$g(X, L) := 1 - \chi_{n-1}(X, L)$: the sectional genus of (X, L)

Castelnuovo varieties

Theorem (Fujita)

Let X be an n-dimensional complex projective variety and let L be a line bundle on X. Assume that L is basepoint free and the morphism defined by L is birational on its image. Then one has

$$g(X,L) \leq m\Delta(X,L) - \frac{1}{2}m(m-1)(L^n - \Delta(X,L) - 1),$$

where m is the largest integer such that $(h^0(L) - n - 1)m \leq L^n - 1$ and $\Delta(X,L) = L^n + n - h^0(L)$, which is called the Δ-genus.

Definition

A polarized variety is called Castelnuovo (in the sense of Fujita) if it satisfies the assumption of the above theorem and $g(X,L)$ achieves the upper bound.

Ehrhart theory

$\mathcal{P} \subset \mathbb{R}^n$: a full-dimensional lattice polytope
$m\mathcal{P} = \{n\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$: the mth dilated polytope of $\mathcal{P}$
$L_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^n|$: the Ehrhart polynomial of $\mathcal{P}$

Theorem (Ehrhart)

$L_{\mathcal{P}}(m)$ is a polynomial in m of degree n.

$$\sum_{m \geq 0} L_{\mathcal{P}}(m) t^m = \frac{h^*(\mathcal{P}, t)}{(1-t)^{n+1}}$$

Remark

- *$h^*(\mathcal{P}, t) := \sum_{i \geq 0} h_i^* t^i$ is a polynomial in t of degree at most n with nonnegative integer coefficients, called the h^*-polynomial of $\mathcal{P}$.*
- *$h_0^* = 1, h_1^* = |\mathcal{P} \cap \mathbb{Z}^n| - (n+1)$ and $h_n^* = |\mathrm{int}(\mathcal{P}) \cap \mathbb{Z}^n|$.*
- *$h_0^* + \cdots + h_n^*$ equals the normalized volume of $\mathcal{P}$.*

Translation into terms of lattice polytopes

$\mathcal{P} \subset \mathbb{R}^n$: a full-dimensional lattice polytope
(X, L) : the associated polarized toric variety of $\mathcal{P}$

- $L^n = h_0^* + \cdots + h_n^*$.
- $h^0(L) = |\mathcal{P} \cap \mathbb{Z}^n| = h_1^* + (n+1)$.
- $g(X, L) = \sum_{i=1}^n (i-1) h_i^*$.
- $\Delta(X, L) = h_2^* + \cdots + h_n^*$.

Remark
L is basepoint free.

Definition
A lattice polytope $\mathcal{P} \subset \mathbb{R}^n$ is called spanning if every lattice point in $\mathbb{Z}^n$ is an affine integer combination of lattice points in $\mathcal{P}$.

Lemma
$\mathcal{P}$ is spanning if and only if the morphism defined by L is birational on its image.

Lattice polytope version of Fujita's theorem

Theorem (Fujita)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a spanning lattice polytope. Then one has

$$\sum_{i=1}^{n}(i-1)h_i^* \leq m(h_2^* + \cdots + h_s^*) - \frac{m(m-1)}{2}h_1^*,$$

where m is the largest integer such that $h_1^* m \leq h_1^* + \cdots + h_n^*$.

We call a lattice polytope Castelnuovo if the associated polarized toric variety is Castelnuovo.

Problem

Characterize Castelnuovo polytopes in terms of lattice polytopes.

Castelnuovo polytopes with interior lattice points

Theorem (Hibi's lower bound theorem)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope with interior lattice points, that is, $h_n^* \neq 0$. Then for any $2 \leq i \leq n-1$, one has $h_1^* \leq h_i^*$.

Theorem (Kawaguchi)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope with interior lattice points. Then $\mathcal{P}$ is Castelnuovo if and only if for any $2 \leq i \leq n-1$, one has $h_1^* = h_i^*$.

Comment to Kawaguchi's proof

Unfortunately, the Kawaguchi's proof did not complete. Indeed, Castelnuovo polytopes are always spanning. However, he missed the spanning assumption. Fortunately, we can complete his proof by the following proposiiton.

Proposition

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope with interior lattice points. If for any $2 \leq i \leq n-1$, $h_1^* = h_i^*$, then $\mathcal{P}$ has a unimodular triangulation. In particular, $\mathcal{P}$ is spanning.

General Castelnuovo polytopes

For a lattice polytope $\mathcal{P}$, we denote $\deg(\mathcal{P})$ the degree of $h^*(\mathcal{P}, t)$.

Theorem (Hofscheier–Katthän–Nill)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a spanning lattice polytope. Then for any $2 \leq i \leq \deg(\mathcal{P}) - 1$, one has $h_1^* \leq h_i^*$.

Theorem (T)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope. Then $\mathcal{P}$ is Castelnuovo if and only if $\mathcal{P}$ is spanning and for any $2 \leq i \leq \deg(\mathcal{P}) - 1$, one has $h_1^* = h_i^*$ and $h_1^* \geq h_{\deg(\mathcal{P})}^*$.

Remark

If $\mathcal{P}$ has an interior lattice point, then one has $h_1^* > h_{\deg(\mathcal{P})}^*$. Therefore, we can get the Kawaguchi's result as a corollary.

Examples

The spanning assumption and the condition $h^*_1 \geq h^*_{\deg(\mathcal{P})}$ are necessary.

Example

Let $\mathcal{P} \subset \mathbb{R}^4$ be the lattice polytope which is the convex hull of

$$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,0,2),(1,0,-1,0).$$

Then one has $h^*(\mathcal{P},t) = 1+t+t^2+t^3$. However, $\mathcal{P}$ is not spanning. Therefore, $\mathcal{P}$ is not Castelnuovo.

Example

Let $\mathcal{P} \subset \mathbb{R}^3$ be the lattice polytope which is the convex hull of

$$(0,0,0),(1,0,0),(0,1,0),(1,2,3),(0,0,-1).$$

Then one has $h^*(\mathcal{P},t) = 1+t+2t^2$ and $\mathcal{P}$ is spanning. However, $h^*_1 < h^*_{\deg(\mathcal{P})}$. Therefore, $\mathcal{P}$ is not Castelnuovo.

Integer decomposition property

Definition

We say that a lattice polytope $\mathcal{P} \subset \mathbb{R}^n$ has the integer decomposition property or is IDP if for any integer $k \geq 1$ and for any lattice point $\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^n$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{P} \cap \mathbb{Z}^n$ such that

$$\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_n.$$

Conjecture (Oda Conjecture)

Every smooth polytope is IDP.

Remark

Smooth polytopes and IDP polytopes are spanning.

Application

Theorem (Fujita)

Let (X, L) be a Castelnuovo variety. Then L is normally generated.

Corollary

Every Castelnuovo polytope is IDP.

Theorem (T)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a spanning lattice polytope. If for any $2 \leq i \leq \deg(\mathcal{P}) - 1$, $h_1^* = h_i^*$ and $h_1^* \geq h_{\deg(\mathcal{P})}^*$, then $\mathcal{P}$ is IDP.

Corollary

Let $\mathcal{P} \subset \mathbb{R}^n$ be a smooth polytope. If for any $2 \leq i \leq \deg(\mathcal{P}) - 1$, $h_1^* = h_i^*$ and $h_1^* \geq h_{\deg(\mathcal{P})}^*$, then $\mathcal{P}$ is IDP.

The case $\deg(\mathcal{P}) = 2$

Proposition

Let $\mathcal{P} \subset \mathbb{R}^n$ be a lattice polytope with $\deg(\mathcal{P}) = 2$. If $h_1^* \geq h_2^*$, then $\mathcal{P}$ is spanning.

Therefore, we can get the following result as a corollary.

Theorem (Katthän–Yanagawa)

Let $\mathcal{P} \subset \mathbb{R}^n$ be a lattice polytope with $\deg(\mathcal{P}) = 2$. If $h_1^* \geq h_2^*$, then $\mathcal{P}$ is IDP.

Question

Question

Does a Castelnuovo polytope have a unimodular triangulation? (If it has interior lattice points, then yes.)