Stable set polytopes in differential algebra

Combinatorial differential algebra of x^p. arXiv:2102.03182
Joint work with Anna-Laura Sattelberger

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Differential Algebra

Let $\mathbb{C}[x^{(\infty)}, \partial_t] = \mathbb{C}[x, x', x^{(2)}, \cdots]$ and $I_{p,n} := \langle x^p, x^{(n+1)} \rangle^{(\infty)}$.
$(\partial_t(I_{p,n}) \subseteq I_{p,n})$.

$\dim(\mathbb{C}[x^{(\infty)}]/I_{p,n})$

$n = 0;\quad \dim(\mathbb{C}[x^{(\infty)}]/I_{p,0}) = p$

$n = 1;\quad \dim(\mathbb{C}[x^{(\infty)}]/I_{p,1}) = \frac{p^2}{2} + \frac{p}{2}$

$n = 6;\ \dim(\mathbb{C}[x^{(\infty)}]/I_{p,6}) = \frac{17}{315}p^7 + \frac{17}{90}p^6 + \frac{53}{180}p^5 + \frac{19}{72}p^4 + \frac{13}{90}p^3 + \frac{17}{360}p^2 + \frac{1}{140}p.$

Question

For fixed n, Is $\dim(\mathbb{C}[x^{(\infty)}]/I_{p,n})$ a polynomial in p of degree $n+1$?

Jet scheme

Let $X = Spec(k[x]/I)$, an m-jet on X is a k-algebra homomorphism

$$\varphi : k[x]/I \to k[t]/(t^{m+1}).$$

Let $f_1, \ldots, f_k$ generators of I

$$x \longmapsto x_0 + x_1 t + \ldots + x_m t^m$$

$f_i(x_0 + x_1 t + \ldots + x_m t^m) = 0$

$$f_i^0 + f_i^1 t + \ldots + f_i^m t^m = 0$$

The m-jet scheme of X is defined by the ideal generated by f_i^k.

Jet scheme of x^p

Let $X = Spec(\mathbb{C}[x]/\langle x^p \rangle)$ and let $R_n = \mathbb{C}[x_0, \ldots, x_n]$.

$$f_{p,n} = (x_0 + x_1 t + \cdots + x_n t^n)^p.$$

Let $C_{p,n}$ the ideal generated by the coefficients of $f_{p,n}$. Then we have

$$R_n / C_{p,n} \xrightarrow{\cong} \mathbb{C}[x^{(\infty)}]/I_{p,n}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}.$$

This map sends the coefficient of t^k in $f_{p,n}$ to $(x^p)^{(k)}$.

Zobnin's Theorem

Let us consider the reverse lexicographic ordering $\prec$ on $\mathbb{C}[x^{(\infty)}]$, the leading monomial of $(x^p)^{(k)}$ is of the form $(x^{(j)})^a(x^{(j+1)})^{p-a}$.

$n = 4,\ p = 3$: $(x^3)^{(4)} = 3x'^2x'' + 3xx''^2 + 6xx'x^{(3)} + 3x^2x^{(4)}$

Theorem (Zobnin [3])

The family $\{(x^p)^{(k)}\}_k$ is a Gröbner basis of the differential ideal $\langle x^p\rangle^{(\infty)}$ in the ring $\mathbb{C}[x^{(\infty)}]$ w.r.t reverse lexicographic ordering.

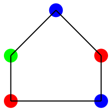
Conclusion:

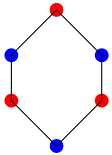
$\dim(\mathbb{C}[x^{(\infty)}]/I_{p,n}) =$
$\#\left\{(u_0,\ldots,u_n) \in (\mathbb{N})^{n+1} \middle| u_i + u_{i+1} \leq p-1 \text{ for all } 0 \leq i \leq n-1\right\}$

Stable set polytope and perfect graph

Definition

Let $G = (V, E)$, we say G is *perfect* if for every subgraph, the chromatic number equals the clique number of that subgraph.
A subset $S \subseteq V$ of vertices is called *stable* if no two elements of S are adjacent.





The *stable set polytope of* G is the $|V|$-dimensional polytope

$$\mathrm{Stab}(G) := \mathrm{conv}\left\{\chi^S \in \mathbb{R}^V \mid S \subseteq V \text{ stable}\right\},$$

where the *incidence vectors* $\chi^S = (\chi^S_v)_{v \in V} \in \mathbb{R}^V$ are defined as

$$\chi^S_v := \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{else.} \end{cases}$$

Fractional stable set polytope

The *fractional stable set polytope of* G is defined as

$$\mathrm{QStab}(G) := \left\{ x \in \mathbb{R}^V \mid 0 \leq x(v) \, \forall v \in V, \ \sum_{v \in Q} x(v) \leq 1 \text{ for all cliques } Q \text{ of } G \right\}.$$

Theorem ([2])

A graph G *is perfect if and only if* $\mathrm{Stab}(G) = \mathrm{QStab}(G)$.

Let $G = (\{0, 1, \ldots, n\}, \{[i, i+1]\}_{i=0,\ldots,n-1})$. We have

$$\mathrm{QStab}(G) = \left\{ (u_0, \ldots, u_n) \in (\mathbb{R}_{\geq 0})^{n+1} \middle| u_i + u_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n-1 \right\}$$

Since G is perfect, $\mathrm{QStab}(G)$ is a lattice polytope whose vertices are binary vectors with no consecutive 1s.

Ehrhart polynomial

Theorem (Ehrhart polynomial)

Let P be a d-dimensional lattice polytope with integer vertices, denote by $L_P(t) := \#(tP \cap \mathbb{Z}^n)$. Then $L_P(t)$ is a polynomial in t of degree d.

Theorem ([1])

$\dim(\mathbb{C}[x^{(\infty)}]/I_{p,n})$ *is the Ehrhart polynomial of* $\mathrm{QStab}(G)$ *computed at* $p-1$.

Two-dimensional case

Consider $\mathbb{C}[x^{(\infty,\infty)}; \partial_t, \partial_s]$ and $I_{p,(m,n)} = \langle x^p, x^{(m+1,0)}, x^{(0,n+1)} \rangle^{(\infty,\infty)}$. Denote by $R_{m,n}$ the polynomial ring in the $(m+1)(n+1)$ many variables $\{x_{k,\ell}\}_{0\leq k\leq m, 0\leq \ell\leq n}$ and let $f_{p,(m,n)}$ be as follow

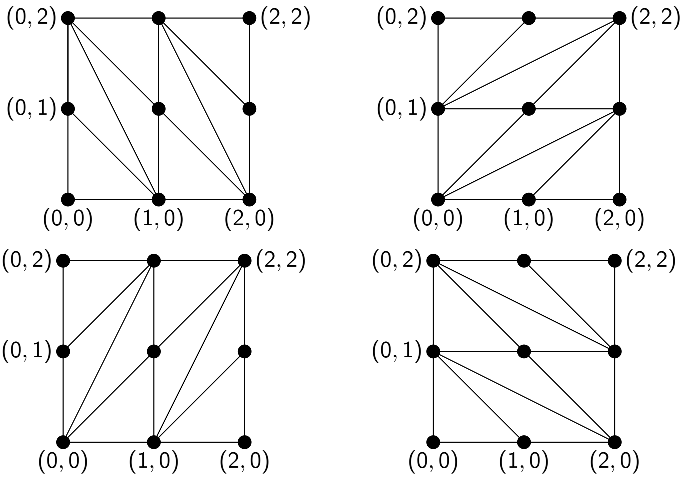
$$f_{p,(m,n)} := \left(\sum_{k=0}^{m} \sum_{\ell=0}^{n} x_{k,\ell} t^k s^\ell \right)^p \in R_{m,n}[s,t].$$

Let $C_{p,(m,n)} \lhd R_{m,n}$ denote the ideal generated by the the coefficients of $f_{p,(m,n)}$. Then

$$R_{m,n}/C_{p,(m,n)} \xrightarrow{\cong} \mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m,n)}, \quad x_{k,\ell} \mapsto \frac{1}{k!\ell!} \cdot x^{(k,\ell)}.$$

$m = n = 2$

There are 64 regular unimodular triangulations of the 2×2-square in total, four of which give rise to a Gröbner basis.

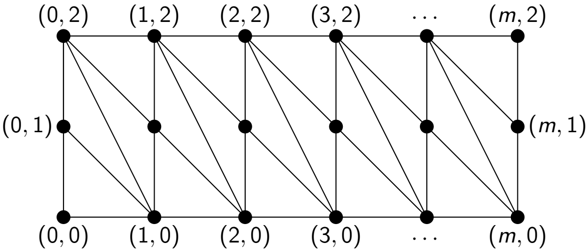


$n = 2$, m arbitrary

Consider the reverse lexicographic ordering on $R_{m,2}$ where the variables are ordered as $x_{00} \prec x_{01} \prec x_{02} \prec x_{10} \prec x_{11} \cdots$.

Theorem ([1])

The leading monomial of $(x^p)^{(k,\ell)}$ *is supported on the triangles of* $T_{m,2}$ *below. Moreover the family* $\{(x^p)^{(k,\ell)}\}_{k \leq mp, \ell \leq 2p}$ *is a Gröbner basis of* $I_{p,(m,2)}$.



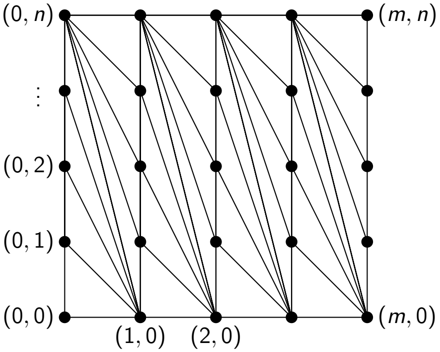
$\dim(\mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m,2)})$

$$\mathrm{QStab}(T_{m,2}) = \Big\{(u_{00},\ldots,u_{m2}) \in (\mathbb{R}_{\geq 0})^{3(m+1)} \Big| u_{k_1,l_1} + u_{k_2,l_2} + u_{k_3,l_3} \leq 1 \text{ for all indices s.t. } \{(k_1,l_1),(k_2,l_2),(k_3,l_3)\} \text{ is a triangle of } T_{m,2}\Big\}$$

Theorem ([1])

$\dim(\mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m,2)})$ *is the Ehrhart polynomial of* $\mathrm{QStab}(T_{m,2})$ *computed at* $p-1$.

$n \geq 3$



Proposition ([1])

For all $n \geq 3$, $\{(x^p)^{(k,\ell)}\}_{k,\ell}$ is NOT a Gröbner basis of $I_{p,(m,n)}$.

R. Ait El Manssour and A.-L. Sattelberger.
Combinatorial differential algebra of x^p, 2021.

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