Permuto-associahedra as deformations of nested permotohedra

Fu Liu

University of California, Davis

(Polytop)ics: Recent advances on polytopes

MPI for Mathematics in the Sciences

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This is joint work with Federico Castillo.

Outline

- Introduction
  - The realization problem
  - General strategy
- Permutohedra, associahedra and permuto-associahedra
- Our construction
  - Nested permutohedra
  - Permuto-associahedra

PART I:

Introduction

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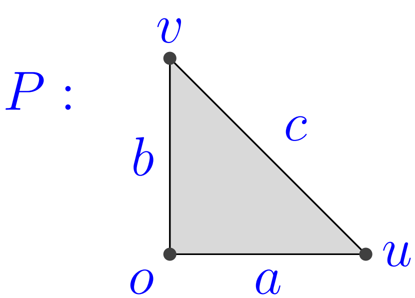
Remark: Both definitions give a geometric embedding of a polytope.

Face poset

Definition. The *(truncated) face poset* of a polytope P, denoted $\mathcal{F}(P)$, is the poset of (nonempty) faces of P ordered by inclusion.

Example.



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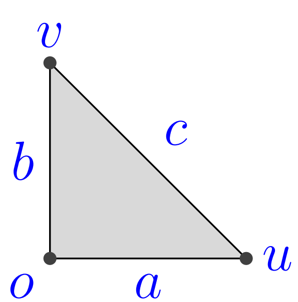

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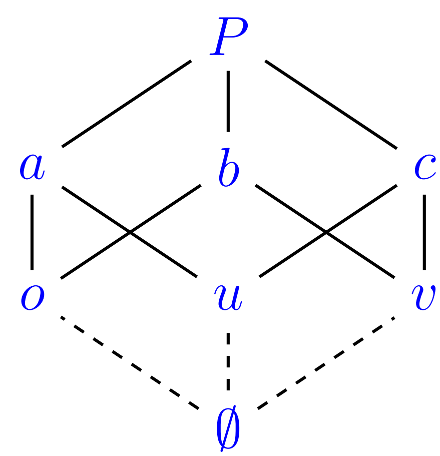

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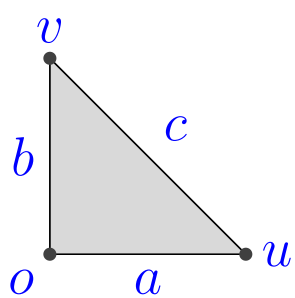

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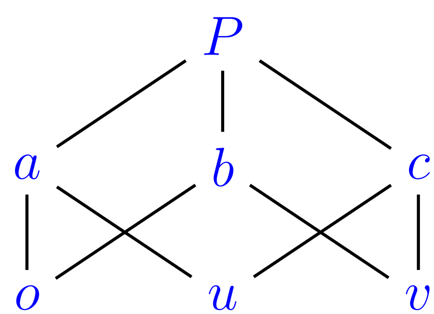

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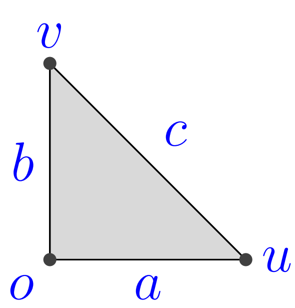

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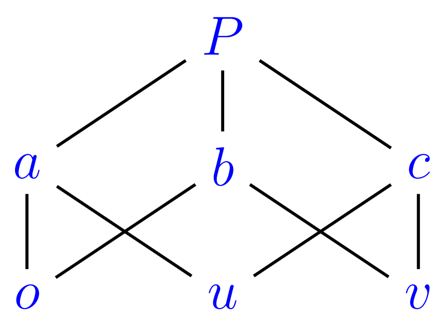

Remark. The face poset $\mathcal{F}(P)$ captures combinatorial properties of the polytope P without specifiying its geometric properties.

Realization problem

Given a “nice” poset $\mathcal{F}$, the following is a classical question to ask:

Does there exist a polytope P such that $\mathcal{F} \cong \mathcal{F}(P)$?

If the answer is yes, we say $\mathcal{F}$ is *realizable*, and such a polytope P a *realization* of $\mathcal{F}$.

Story: Realization of Kapranov's poset

(1) Kapranov:

- defined a poset $\mathcal{K}\Pi_d$ which is a hybrid between the face poset of the permuto-hedron and the associahedron. (This was motivated by MacLane's coherence theorem for associativities and commutativies in monoidal categories.)
- showed that $\mathcal{K}\Pi_d$ is the face poset of a CW-ball.
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- provided a realization for $\mathcal{K}\Pi_d$,
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(3) We constructed nested permutohedra in our early work, and noticed its connection to the permuto-associahedron which leads to our realization.

General strategy

Question:

How to construct a polytope with a given face poset
or more generally satisfying certain combinatorial properties?

Our strategy for construction:

(1) Construct candidates for the vertex set and the normal fan of the polytope.

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Normal cones and Normal fan

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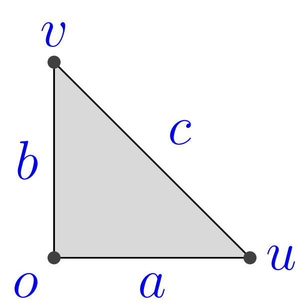

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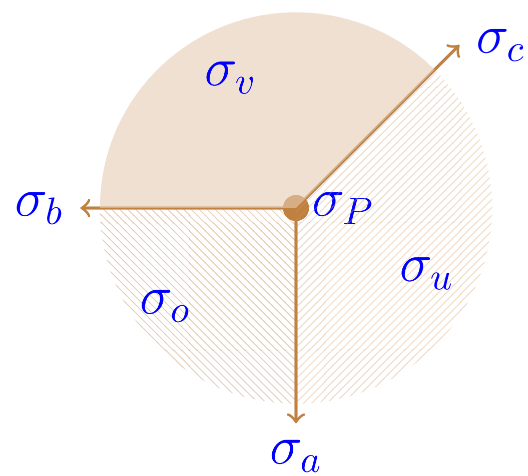

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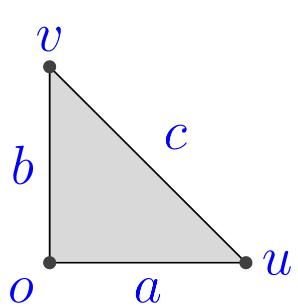

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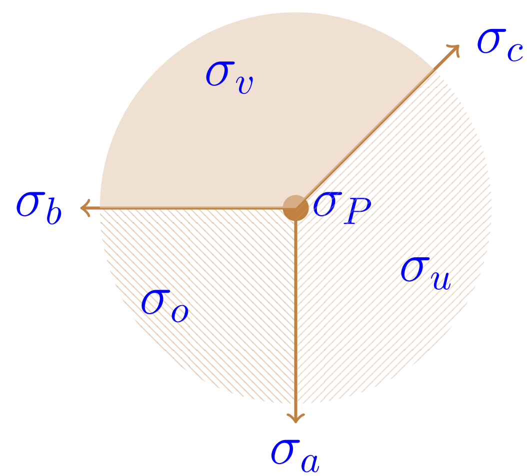

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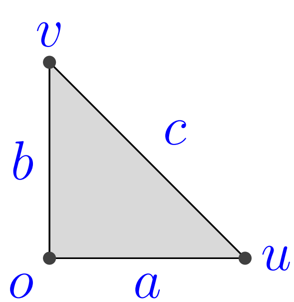

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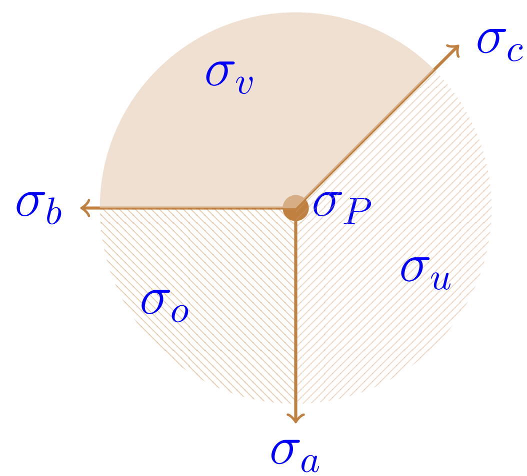

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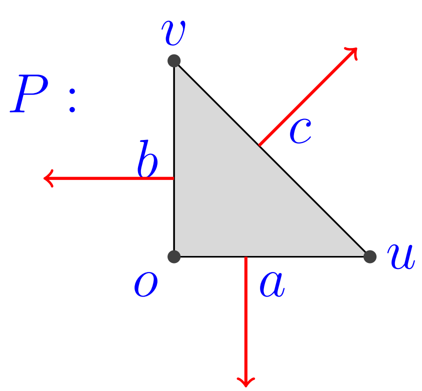

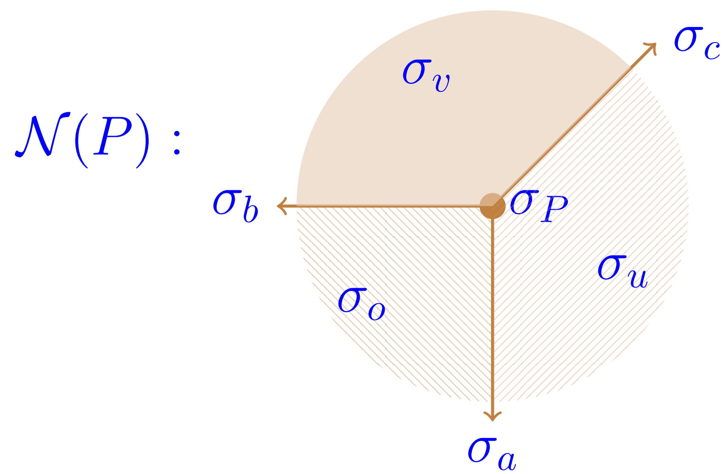

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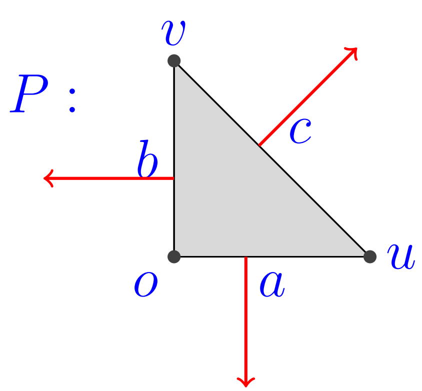

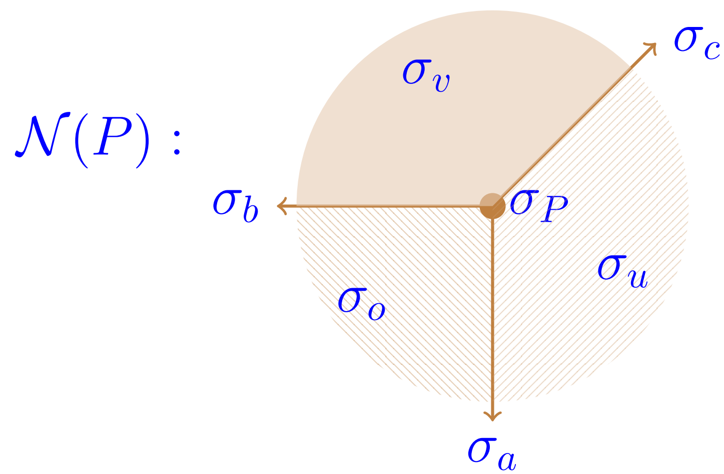

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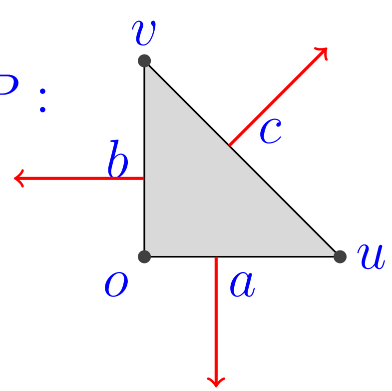

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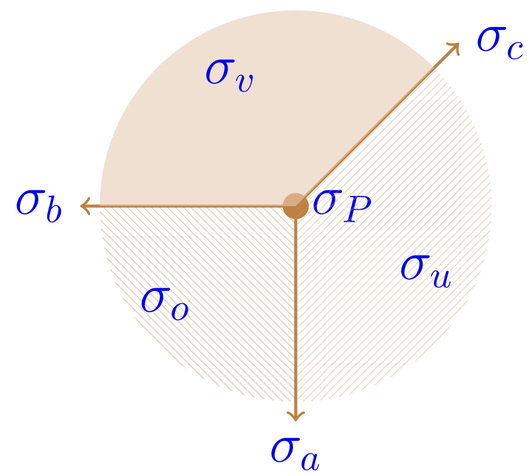

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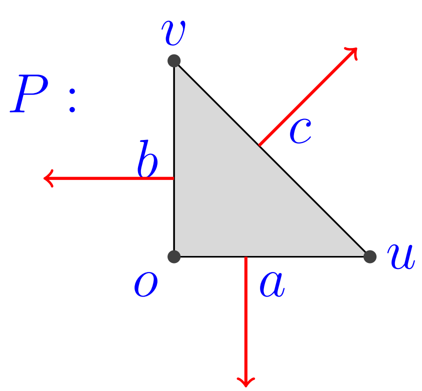

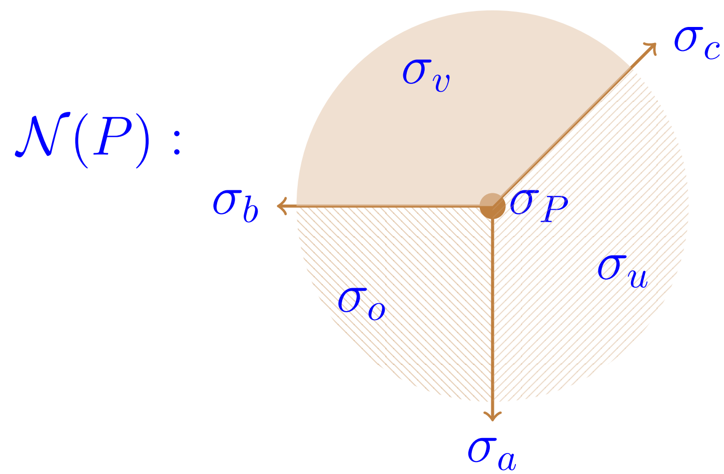

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Therefore, $$\mathcal{F}(P) \cong \mathcal{F}^*(\mathcal{N}(P))$$

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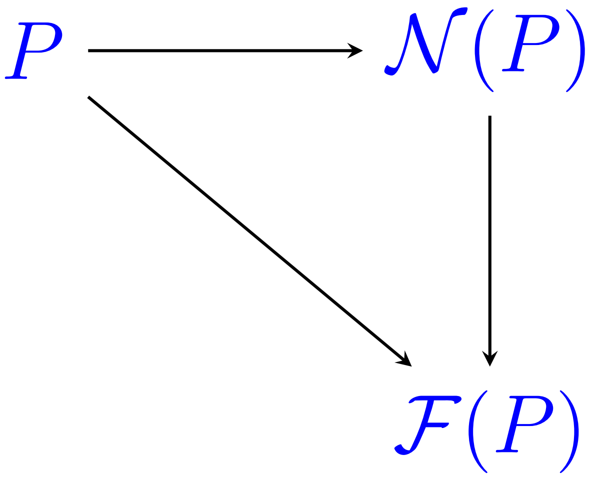

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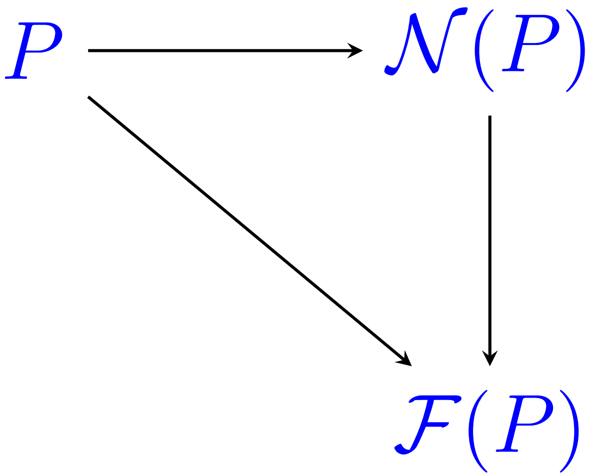
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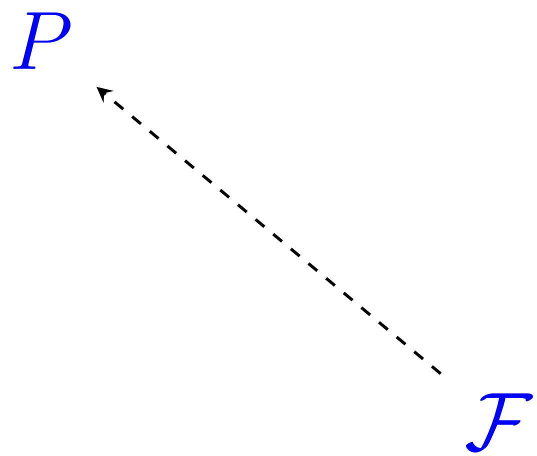
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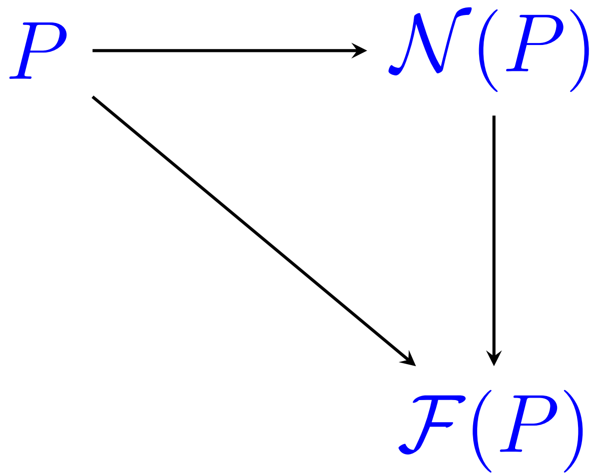
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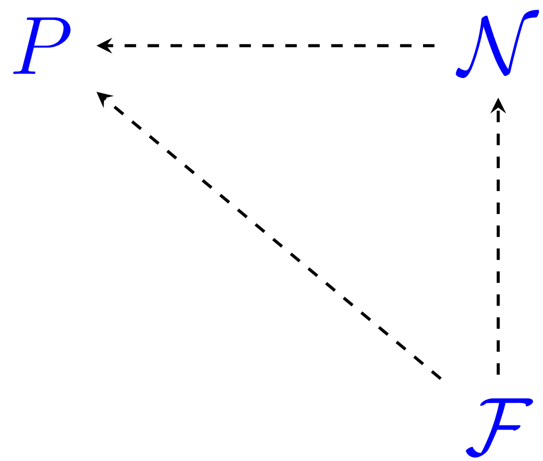
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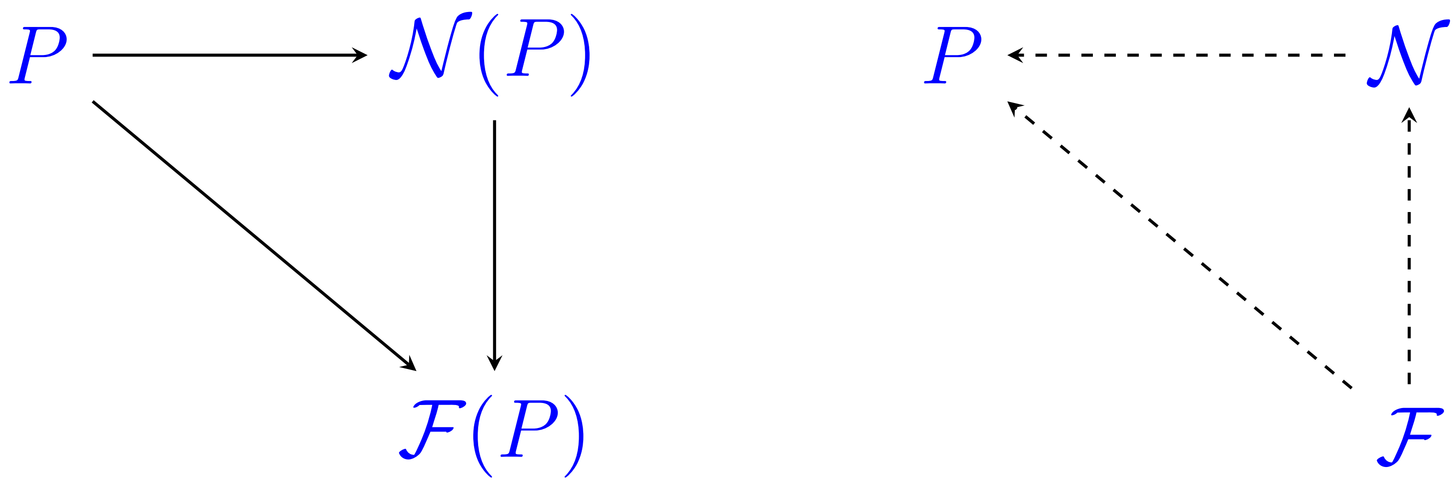

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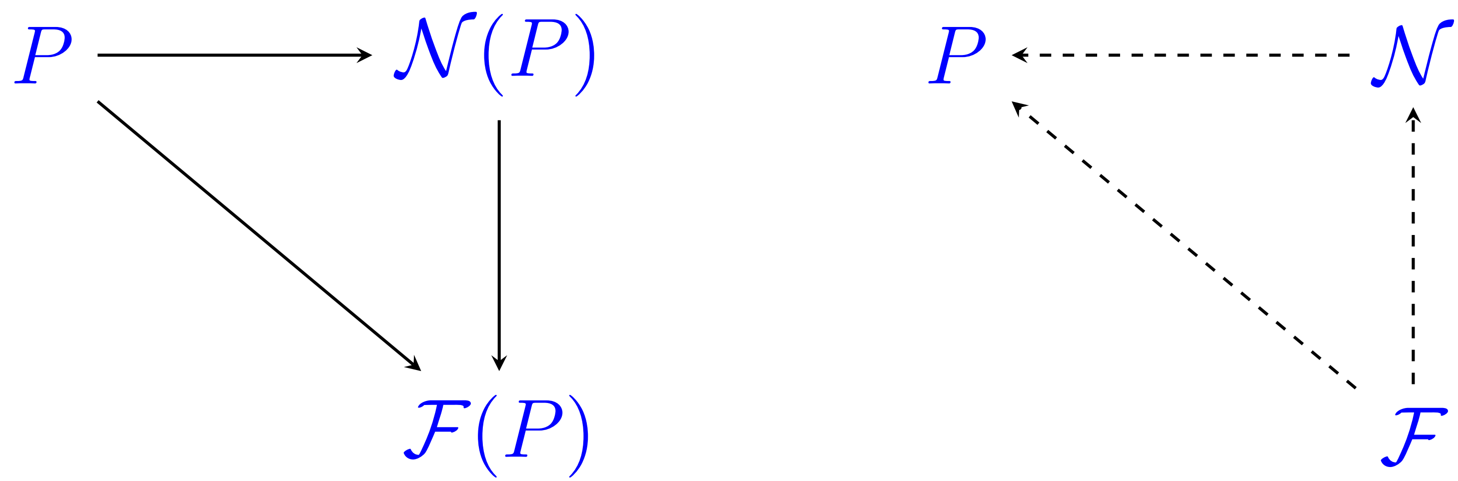

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Note: The **top** dimensional cones in a normal fan are the normal cones at **vertices**, which are minimal elements of the face poset.

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Goal:

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Remark. In addition to knowing its vertex set, normal fan, we can quickly obtain its inequality description.

PART II:

Permutohedra, Associahedra and

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Usual permutohedra

Definition. Given a strictly increasing sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$, for any $\pi \in \mathfrak{S}_{d+1}$, we use the following notation:

$$v_\pi^{\boldsymbol{\alpha}} := \left(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \cdots, \alpha_{\pi(d+1)}\right).$$

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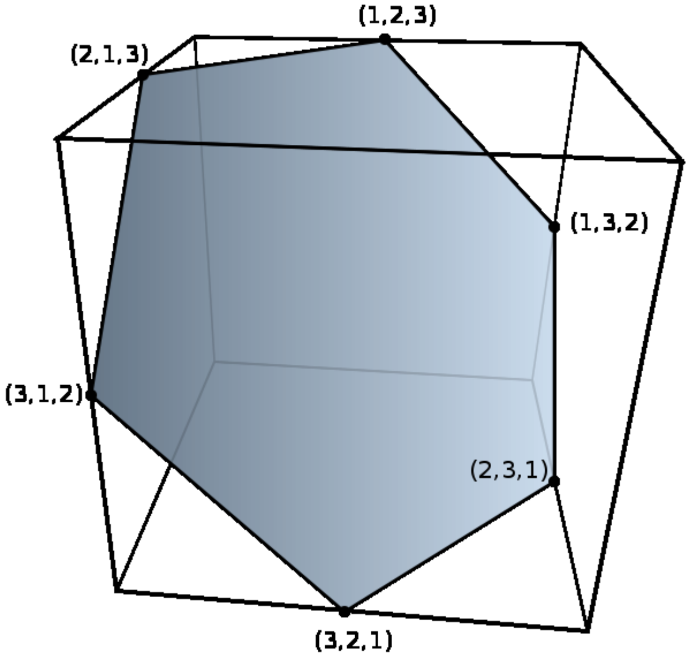
- If $\boldsymbol{\alpha} = (1, 2, \ldots, d, d+1)$, we obtain the *regular permutohedron* Π_d.

Example. Π_2 and Π_3:



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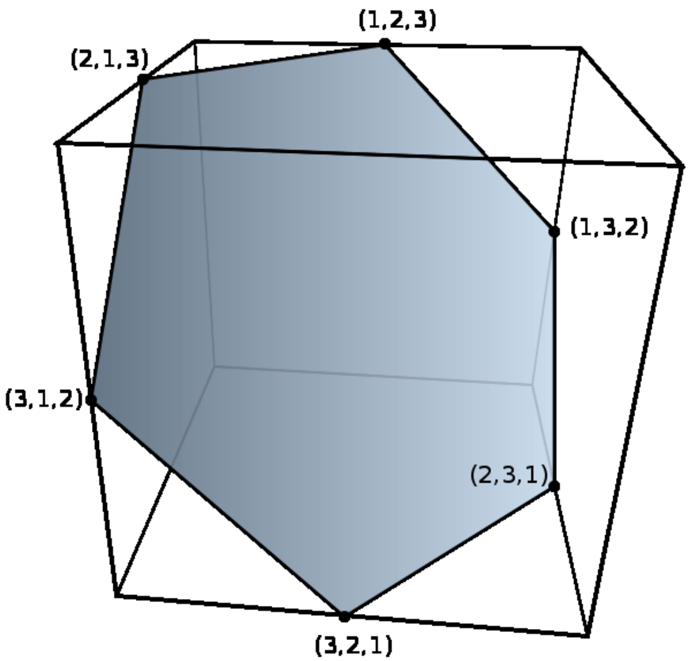

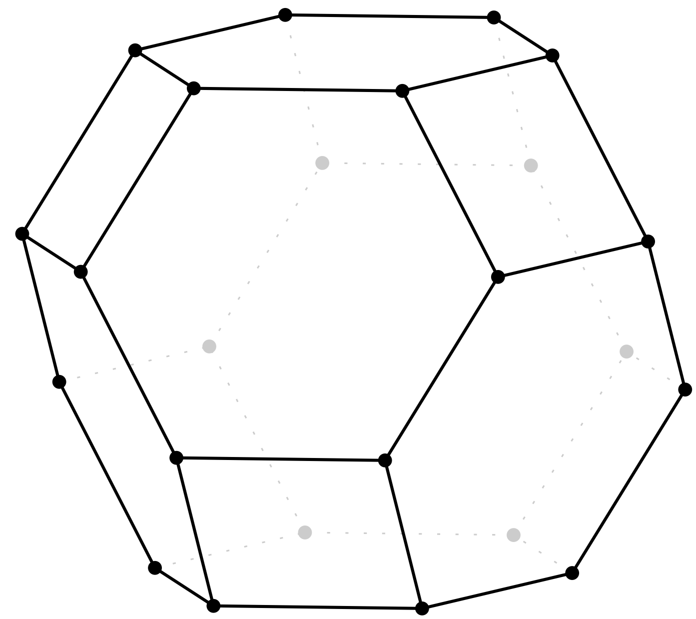

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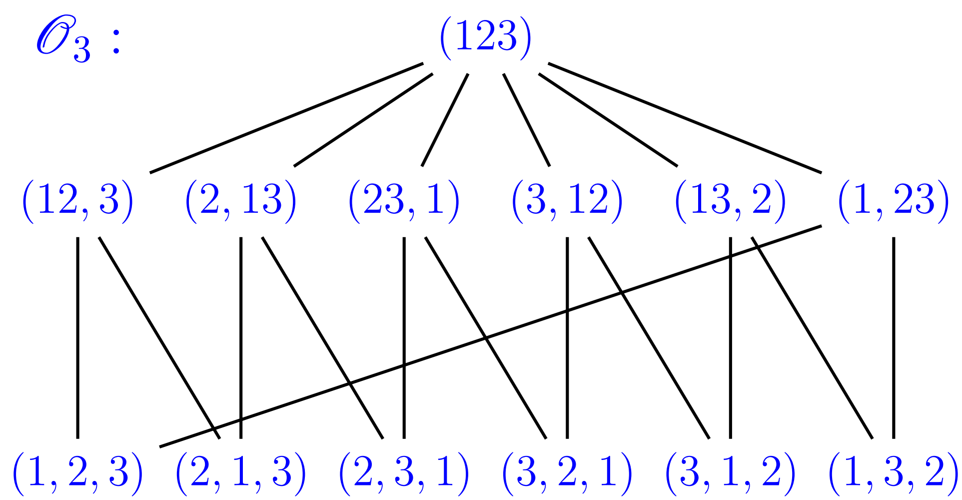

Poset of ordered partitions

Definition. An *ordered (set) partition* of $[d+1]$ is an ordered tuple $\mathcal{S} = (S_1, \cdots, S_k)$ where $S_1, \dots, S_k$ are k disjoint sets whose union is $[d+1]$.

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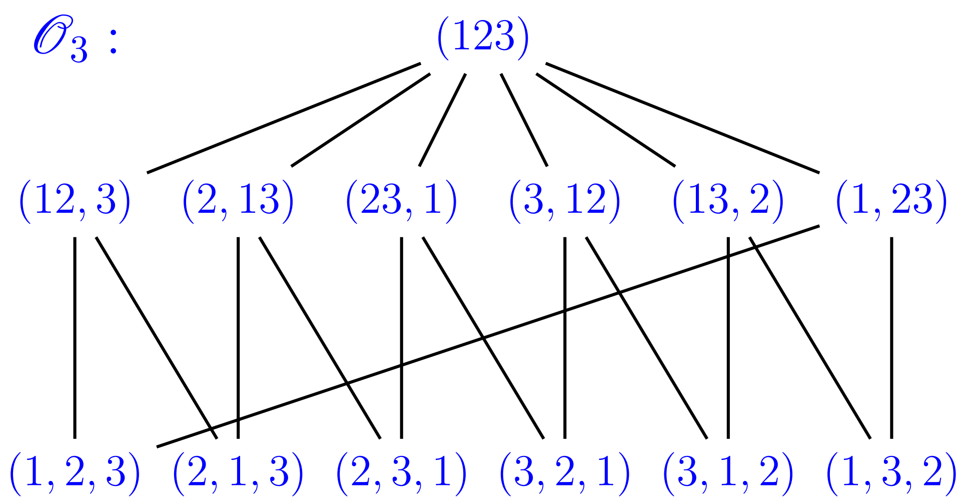

It is well-known that

$$\mathcal{F}(\mathrm{Perm}(\boldsymbol{\alpha})) = \mathscr{O}_{d+1}$$

Associahedra

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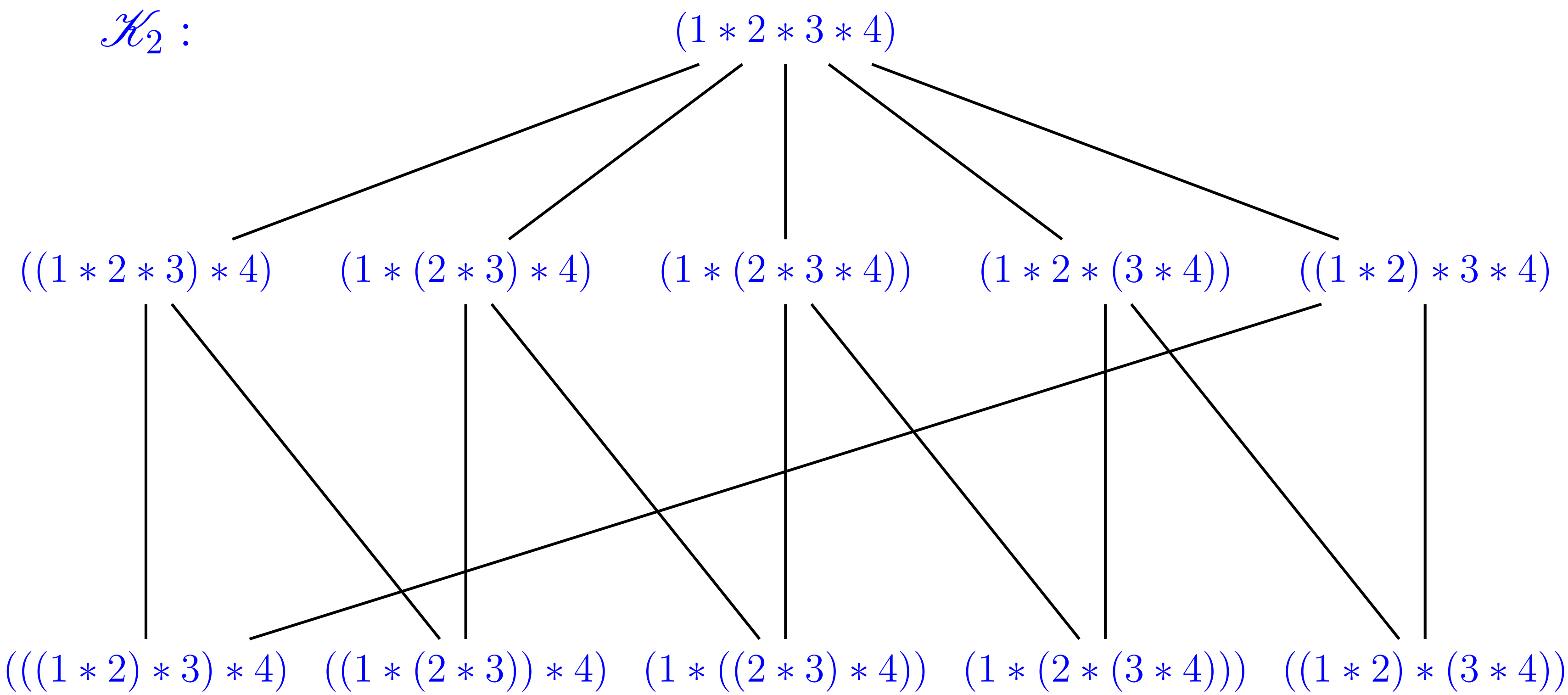

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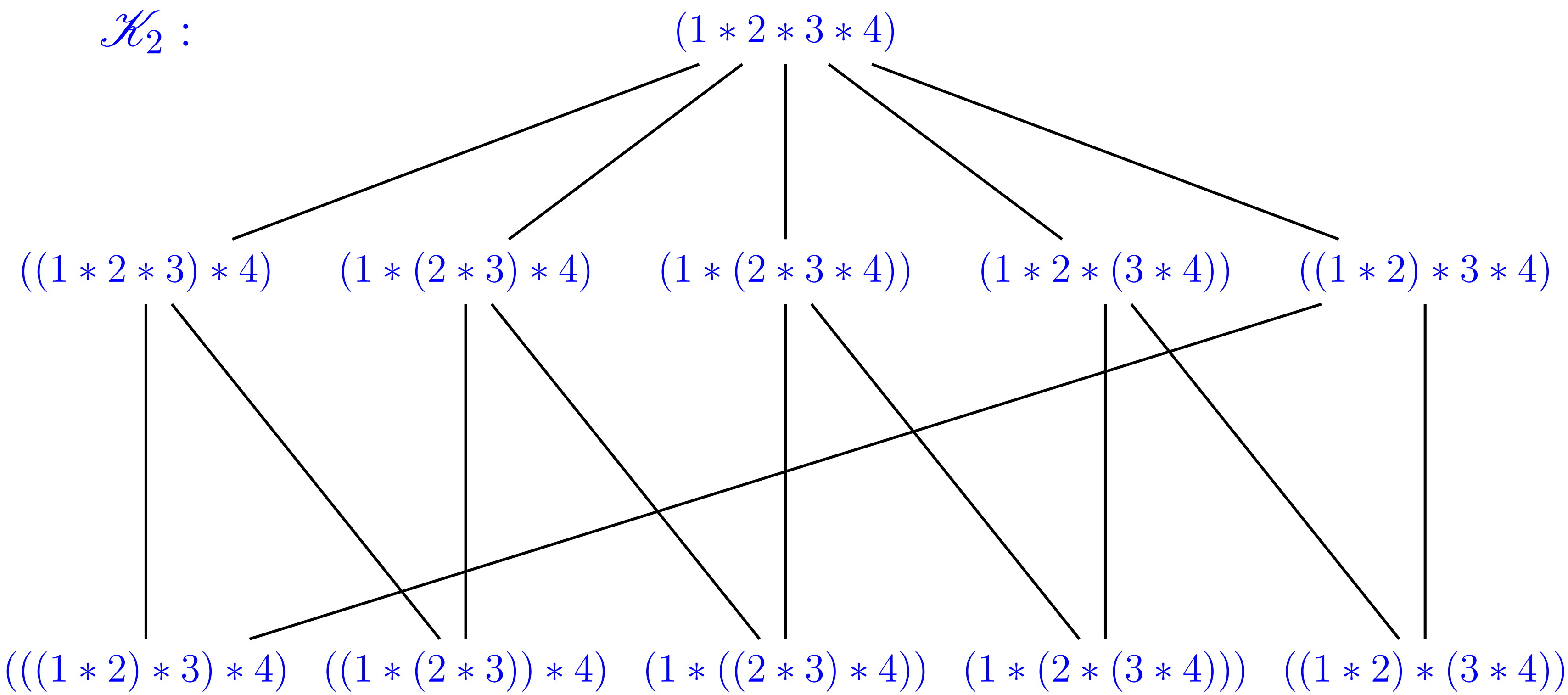

Definition. A d-dimensional *associahedron* is a polytope whose face lattice is $\mathscr{K}_d$.

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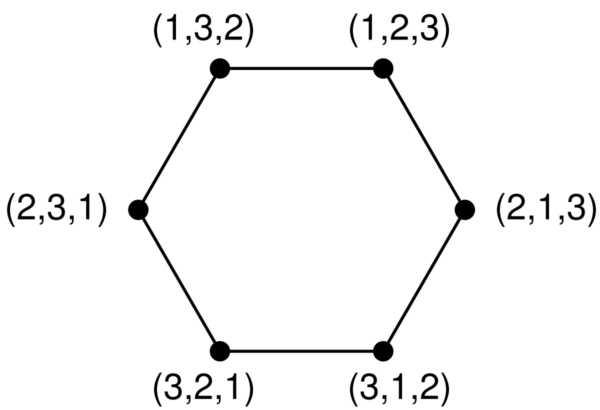

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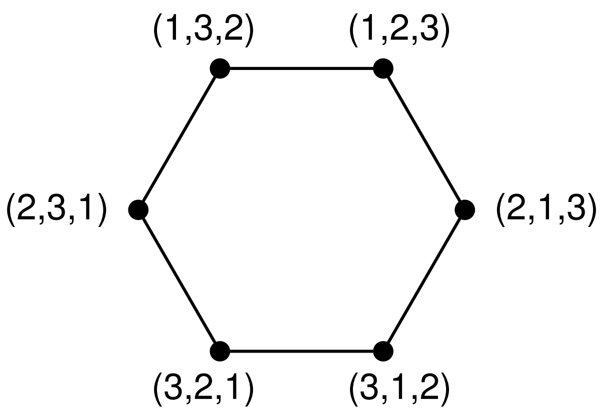

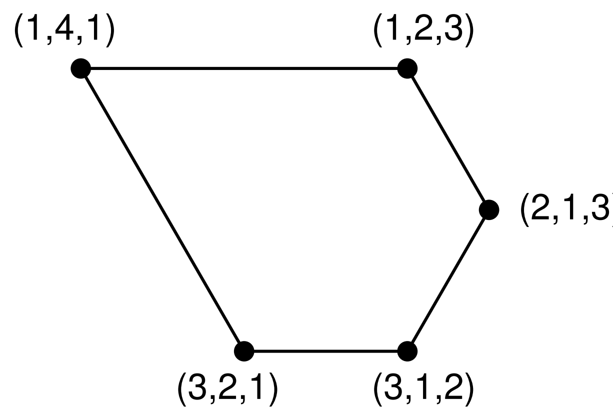

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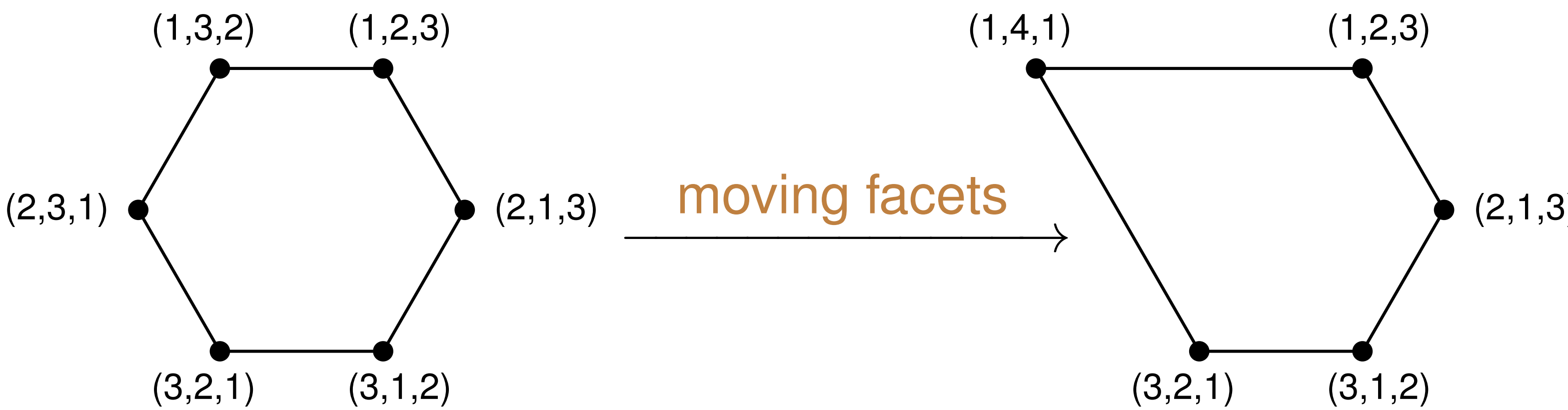

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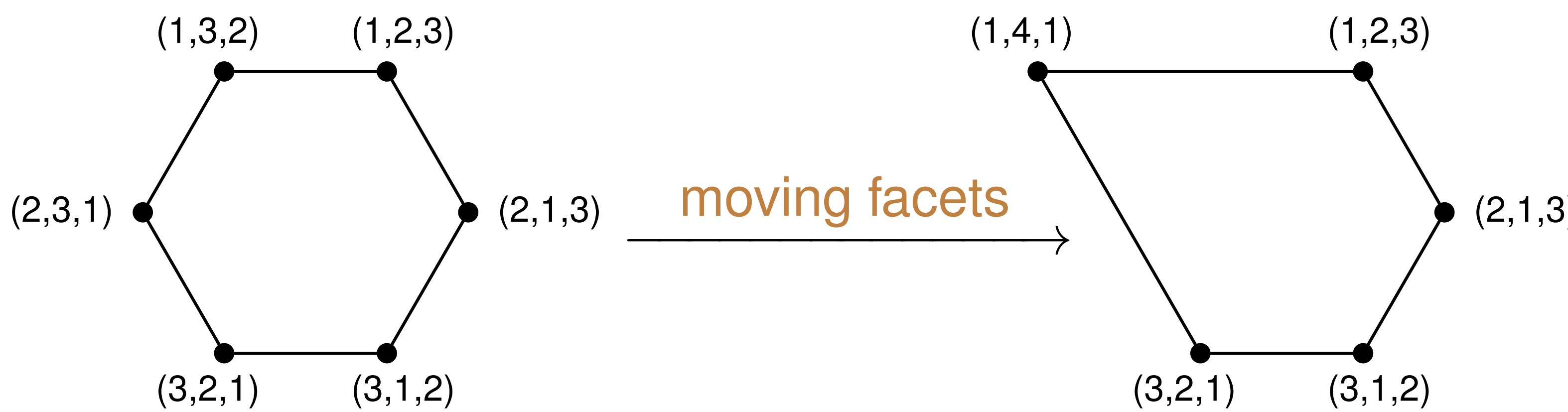

Note: Constructions of associahedra by Shnider-Sternberg, Postnikov, Rote-Santos-Streinu, Hohlweg-Lange, and Buchstaber are all very related to Loday's realization.

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Observation. $\mathcal{K}\Pi_d$ is graded of rank d.

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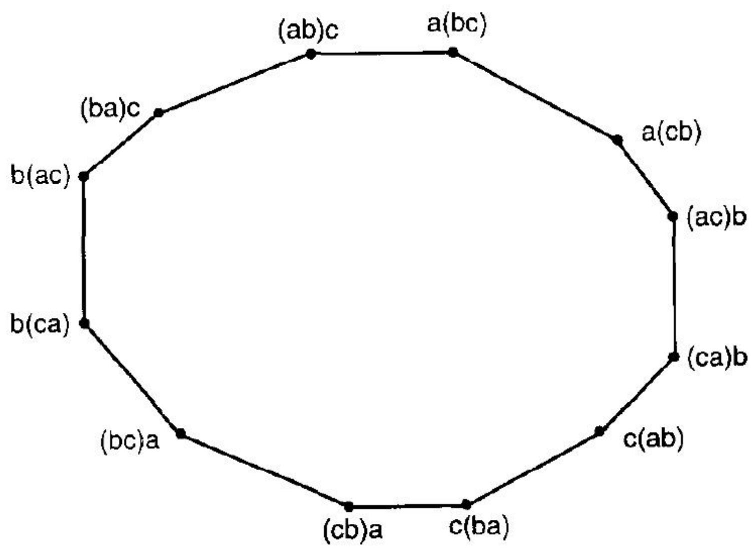

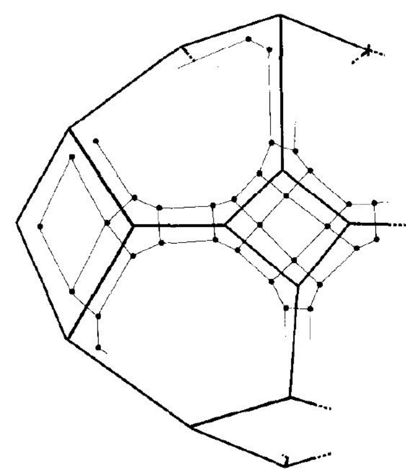

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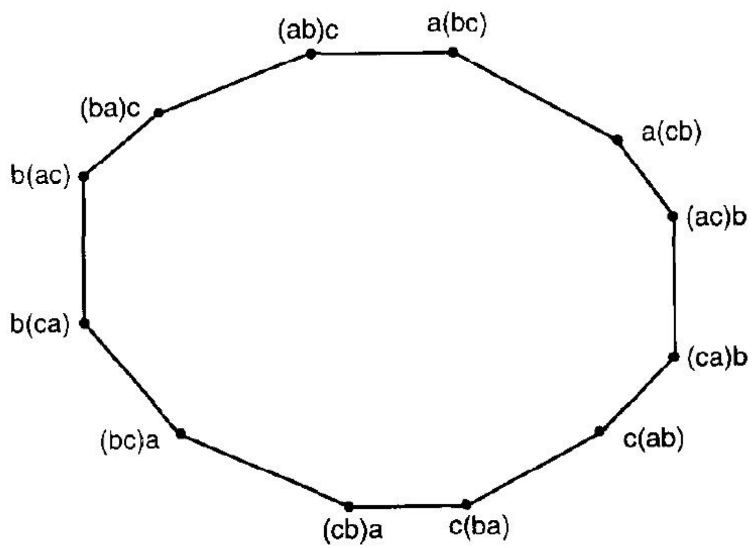
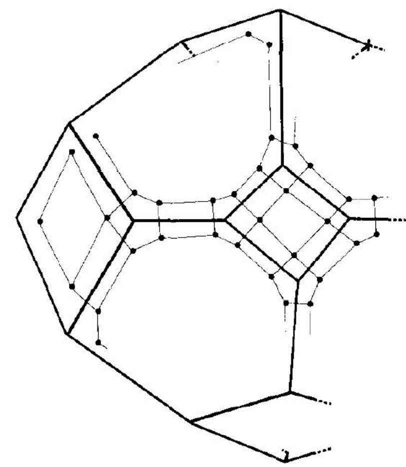
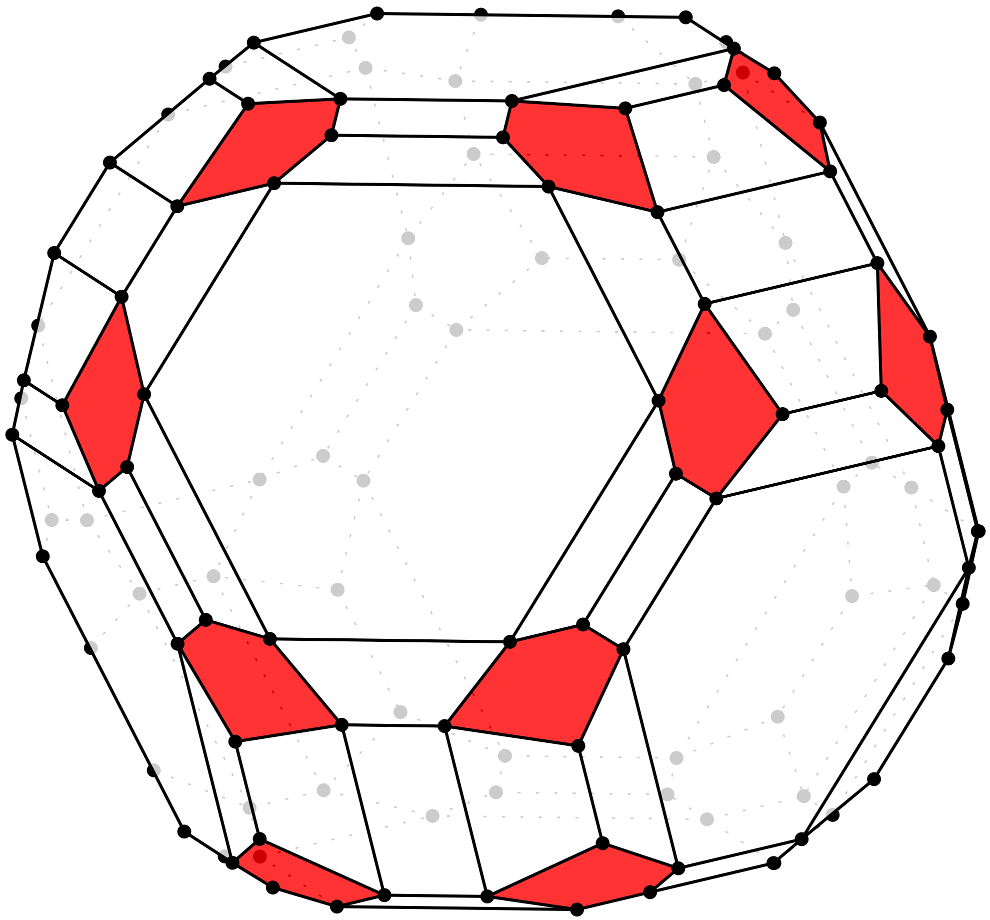




The 2nd picture suggests: put a **small** $(d-1)$-dimensional associahedron at each vertex of a d-dimensional permutohedron.

Failed constructions



Permute a **generic** pentagon. Generally you get 8 edges (instead of 6) coming out of each pentagon.

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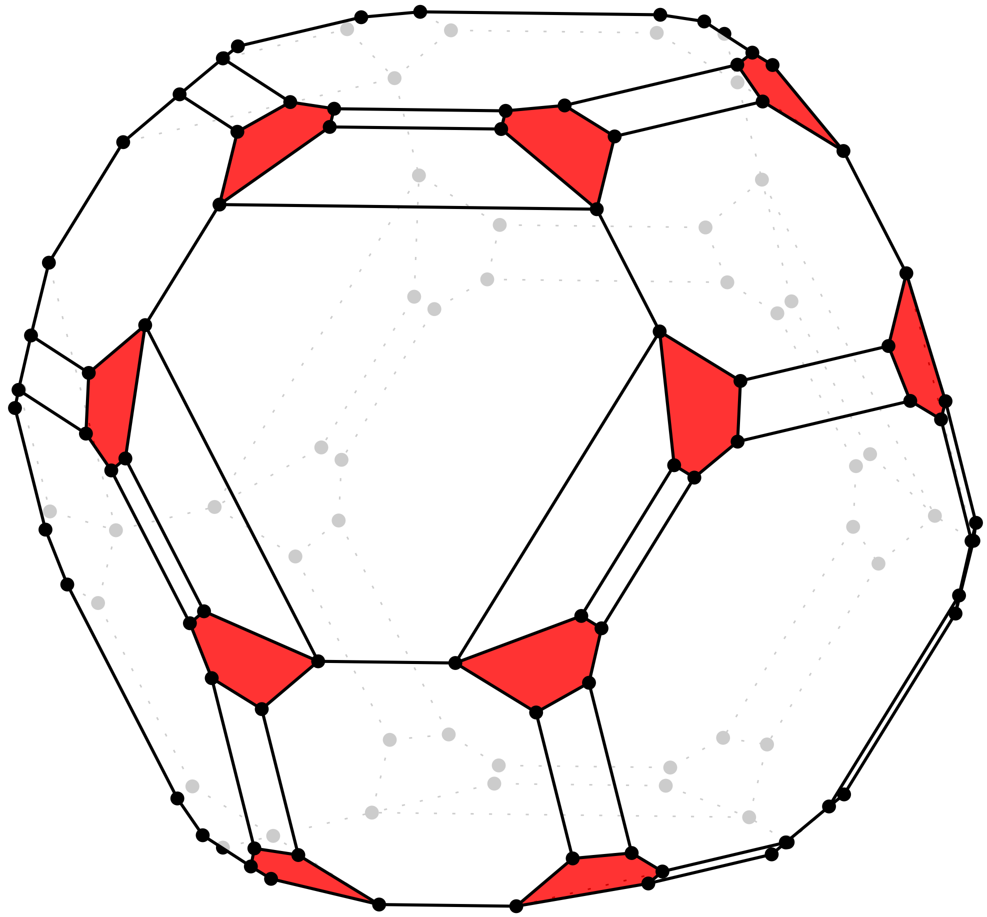

This polytope has the **correct face-vector** but the **wrong combinatorics**.

Reiner-Ziegler's construction

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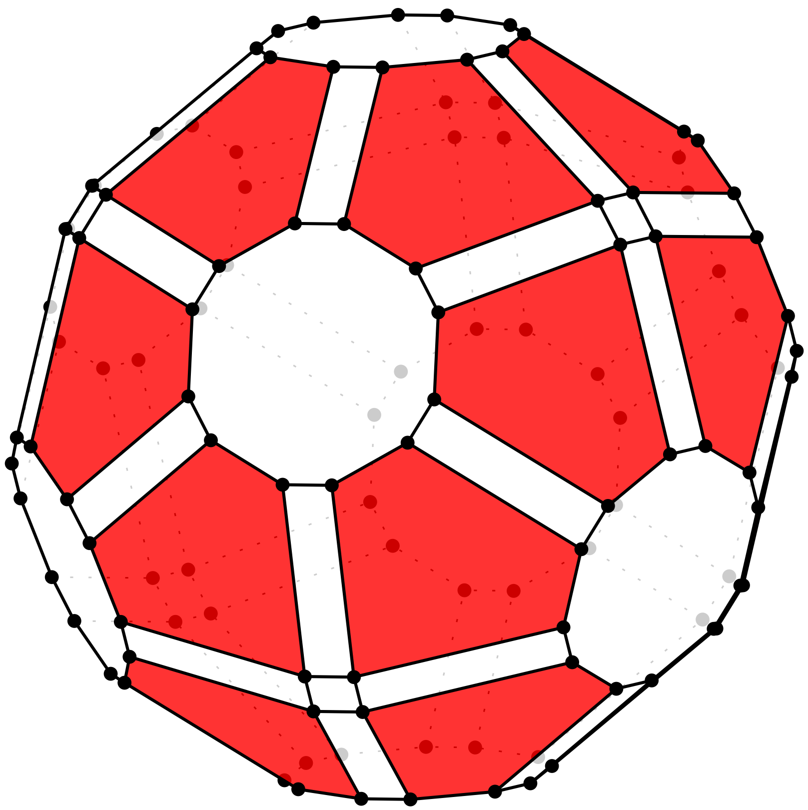


Permute a **projected** GTZ associahedron constructed from a cyclic polygon.

PART III:

Our Construction

- Nested permutohedra
- Permuto-associahedra

Our strategy

Goal:

Construct a polytope P with a given face poset $\mathcal{F}$

or more generally satisfying certain combinatorial properties.

(1) Construction:

(a) Construct vertex set candidate $\{v_i\}$,

(b) Construct candidates for top dimensional cones $\{\sigma_i\}$.

(2) Verify that $\{v_i\}$ and $\{\sigma_i\}$ "match".

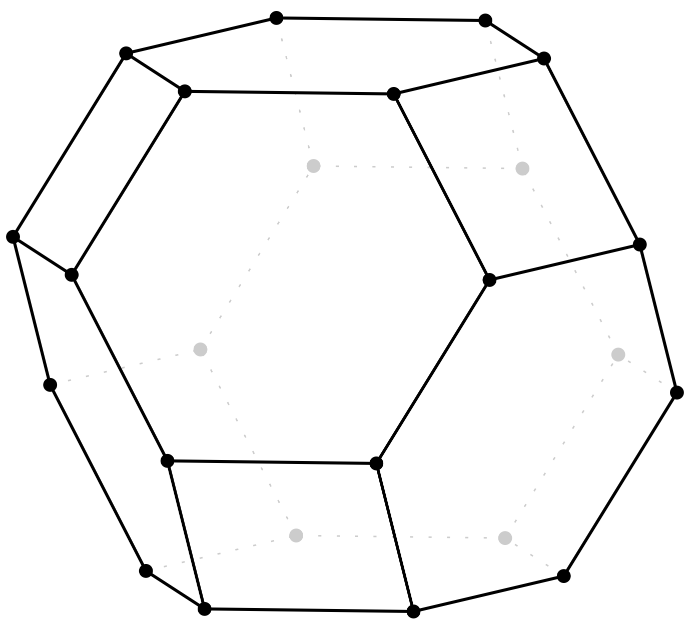
Then conclude $P := \text{conv}\{v_i\}$ is a desired polytope where

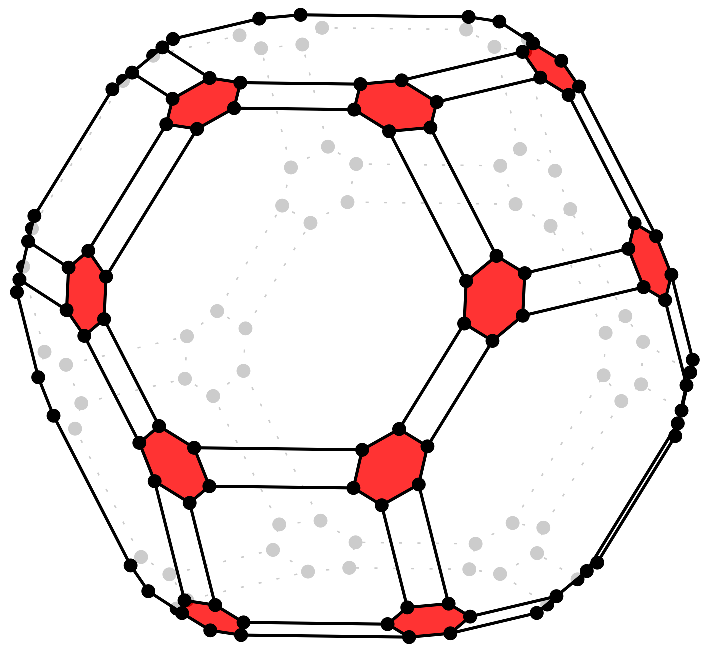
- $\{v_i\}$ is its vertex set and
- $\{\sigma_i\}$ are the top dimensional cones in its normal fan.

We can also obtain an inequality description for P.

Usual nested permutohedra

Definition (Informal)**.** Replace each vertex of a usual permutohedron $\mathrm{Perm}(\boldsymbol{\alpha})$ by a smaller dimension permutohedron $\mathrm{Perm}(\boldsymbol{\beta})$ (in the correct orientation). We obtain the *usual nested permutohedron* $\mathrm{Perm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.





One requirement: Entries in α is suffciently larger than entries in β

Vertex set candidate for Usual N.P.

Recall that $\{v_\pi^{\boldsymbol{\alpha}} : \quad \pi \in \mathfrak{S}_{d+1}\}$ is the vertex set of $\mathrm{Perm}(\boldsymbol{\alpha})$, where

$$v_\pi^{\boldsymbol{\alpha}} := \left(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \cdots, \alpha_{\pi(d+1)}\right) = \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}.$$

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For any $(\pi, \tau) \in \mathfrak{S}_{d+1} \times \mathfrak{S}_d$, we define

$$v_{\pi,\tau}^{(\boldsymbol\alpha,\boldsymbol\beta)} := \underbrace{\sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}}_{v_\pi^{\boldsymbol\alpha}} + \underbrace{\sum_{i=1}^{d} \beta_i \boldsymbol{f}^\pi_{\tau^{-1}(i)}}_{v_\tau^{\boldsymbol\beta} \text{ in correct orientation}},$$

where for any permutation $\pi \in \mathfrak{S}_{d+1}$,

$$\boldsymbol{f}_i^\pi := \boldsymbol{e}_{\pi^{-1}(i+1)} - \boldsymbol{e}_{\pi^{-1}(i)}, \quad \forall 1 \le i \le d.$$

Top dimensional cones for Usual N.P.

Recall that the normal cone of $\mathrm{Perm}(\boldsymbol{\alpha})$ at $v_\pi^{\boldsymbol{\alpha}}$ is:

$$\sigma(\pi) := \{\boldsymbol{w} \in V^* \ : \ w_{\pi^{-1}(1)} \le w_{\pi^{-1}(2)} \le \cdots \le w_{\pi^{-1}(d+1)}\}.$$

is a cone determined by *ordering of coordinates* associated with the permutation $\pi \in \mathfrak{S}_{d+1}$.

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that is,

$$\sigma(\pi,\tau) := \left\{ \boldsymbol{w} \in V^* \ : \ \begin{array}{c} \underbrace{w_{\pi^{-1}(1)} \le \overbrace{w_{\pi^{-1}(2)}}_{\Delta_1} \le w_{\pi^{-1}(3)}}^{\Delta_2} \le \cdots \le \underbrace{w_{\pi^{-1}(d)} \le w_{\pi^{-1}(d+1)}}_{\Delta_d} \\ \Delta_{\tau^{-1}(1)} \le \Delta_{\tau^{-1}(2)} \le \cdots \le \Delta_{\tau^{-1}(d)} \end{array} \right\}.$$

We verify that $\left\{ v_{\pi,\tau}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \right\}$ and $\{\sigma_{\pi,\tau}\}$ "match". Hence:

$$\mathrm{Perm}(\boldsymbol{\alpha},\boldsymbol{\beta}) := \mathrm{conv}\left(v_{\pi,\tau}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \ : \ (\pi,\tau) \in \mathfrak{S}_{d+1} \times \mathfrak{S}_d \right)$$

is the usual nested permutohedron we look for.

Question:

Recall that Loday's associahedron is a deformation of a regular permutohedron.

Can we give a realization of permuto-associahedron as a deformation of a usual nested permutohedron?

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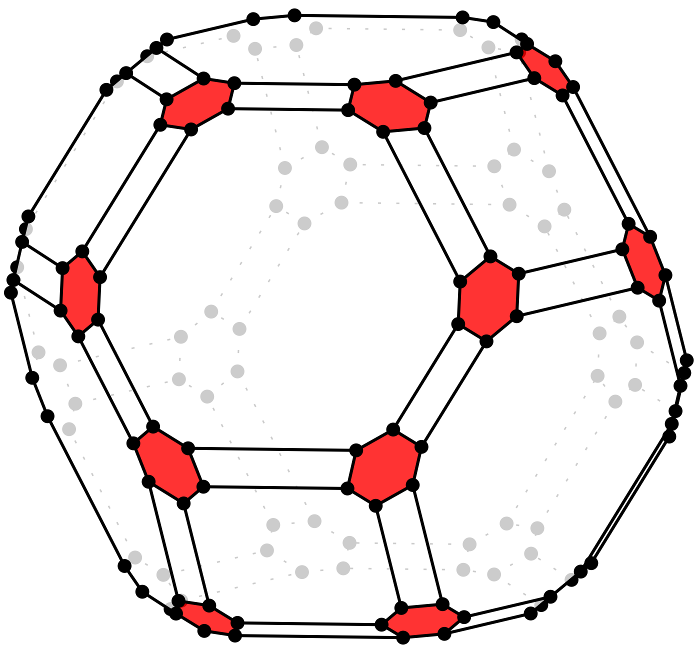
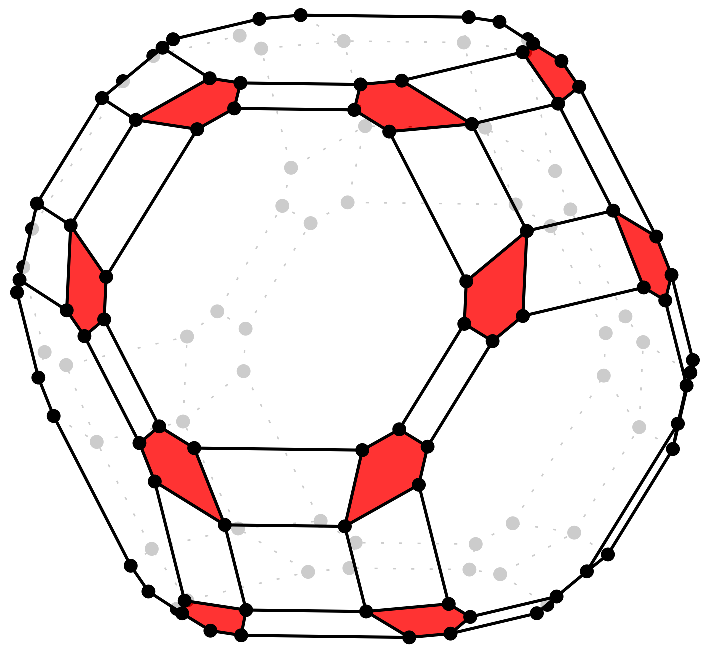
Answer: **Yes!**





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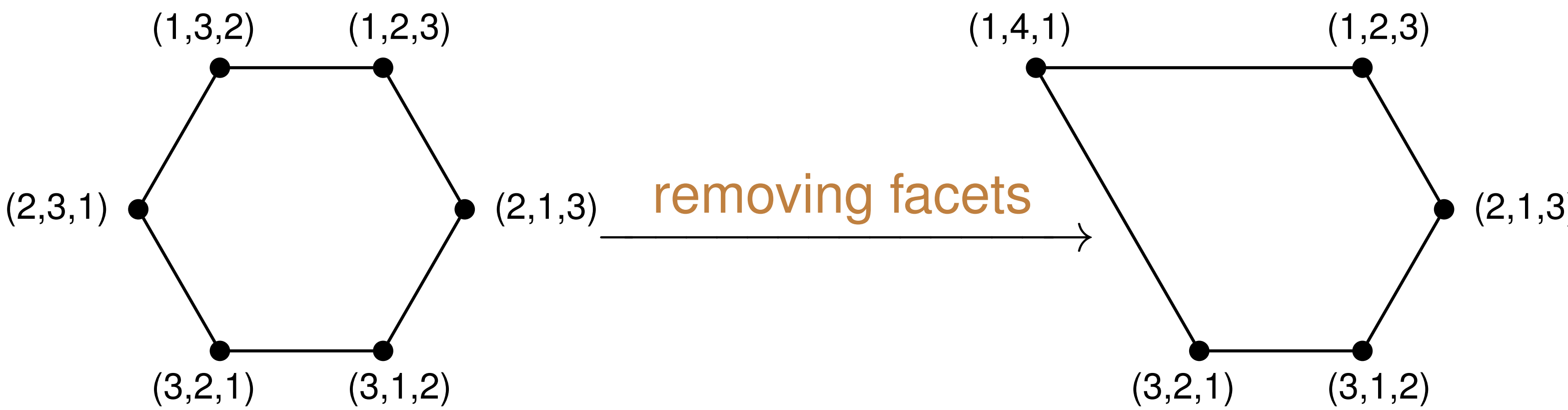
(1) Loday's associahedron actually is a *removohedron*.



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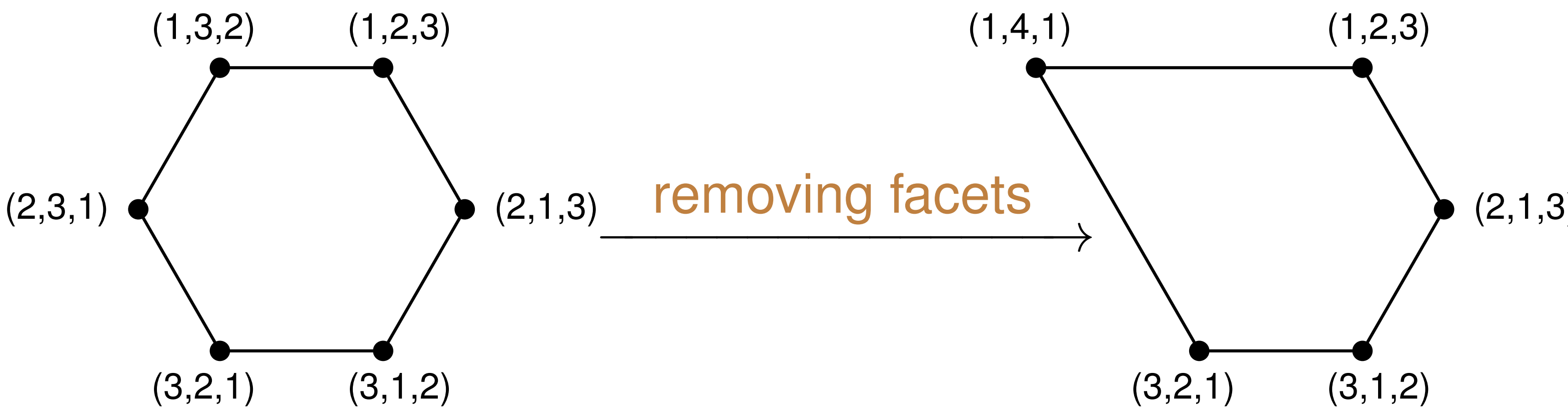

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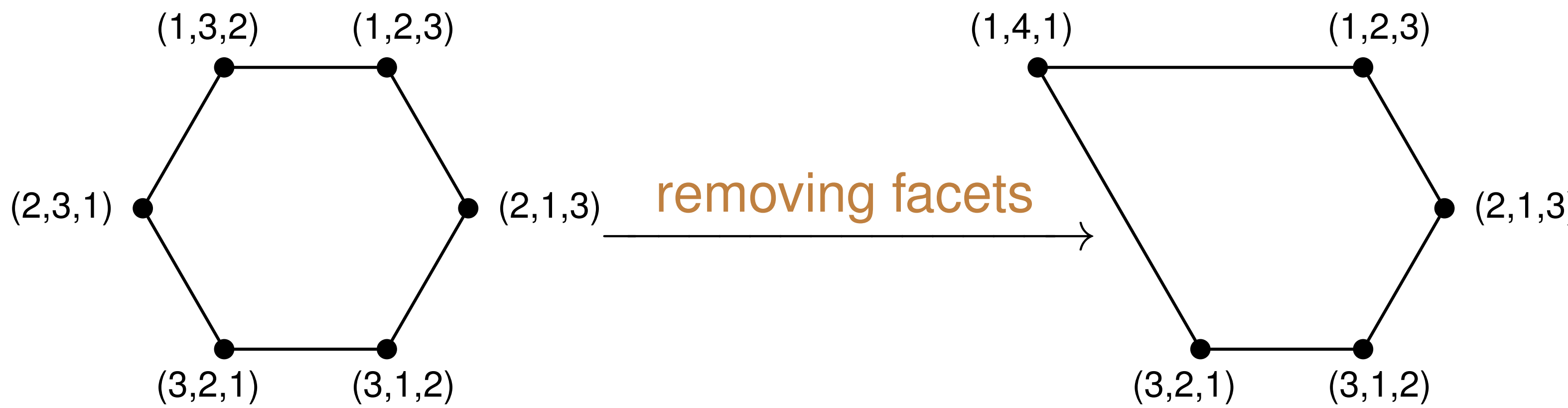

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The construction of $\mathrm{LodayAsso}(\boldsymbol{\beta})$ was constructed using our general strategy. Hence, we have constructed **(i)** its vertex set $\{v_{\mathrm{T}}^{\boldsymbol{\beta}}\}$, and

(ii) its top dimensional normal cones $\{\sigma_{\mathrm{T}}\}$, which induces its normal fan. We call it the *Loday fan*.

Details of our construction

(2) Informally, by replacing each vertex of a usual permutohedron $\mathrm{Perm}(\boldsymbol{\alpha})$ by a smaller dimension permutohedron $\mathrm{LodayAsso}(\boldsymbol{\beta})$ (in the correct orientation). We obtain the *permuto-associahedron* $\mathrm{PermAsso}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

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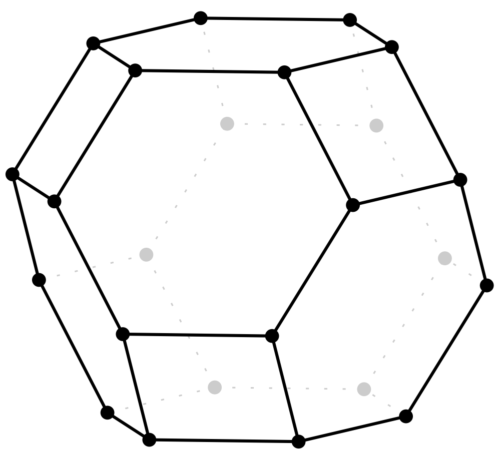
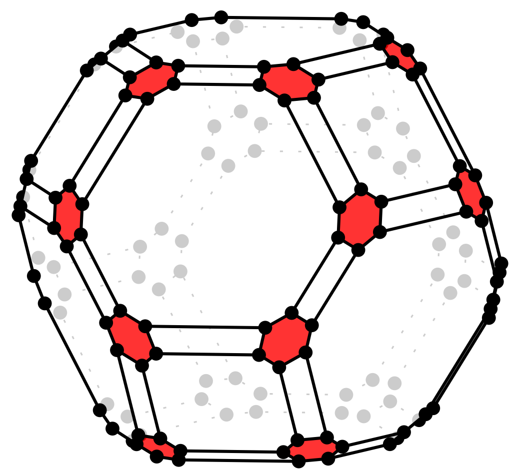
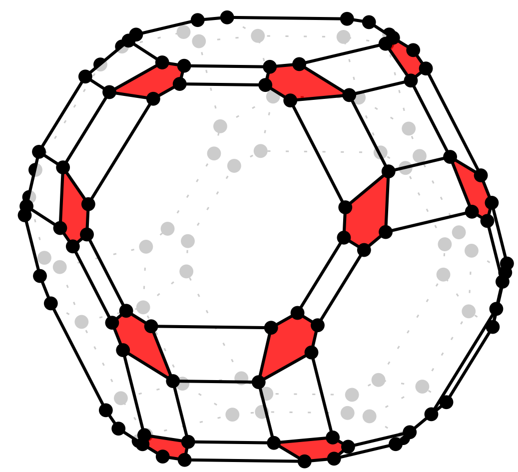

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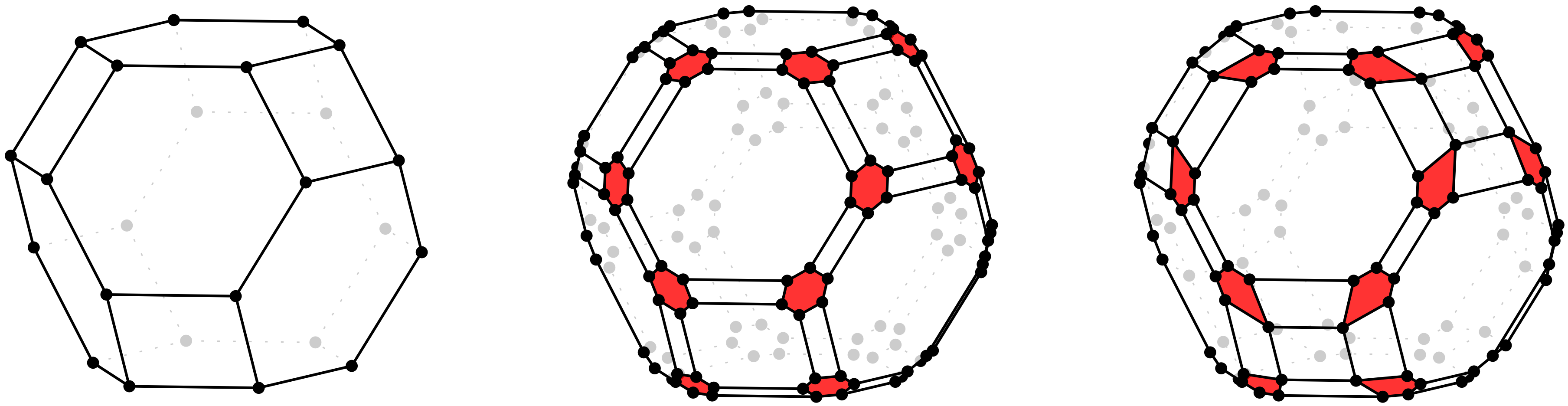

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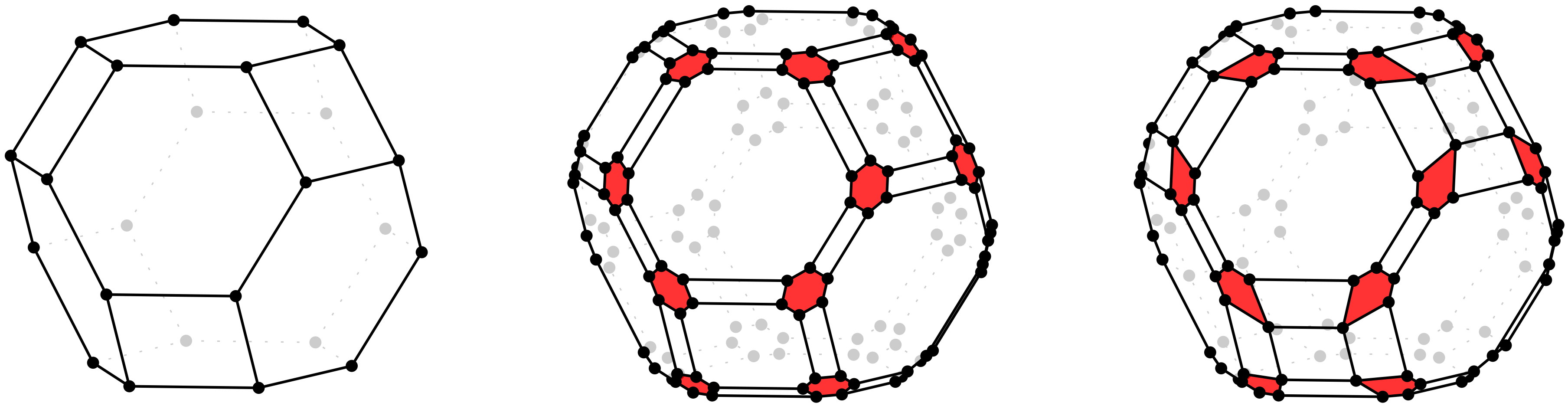

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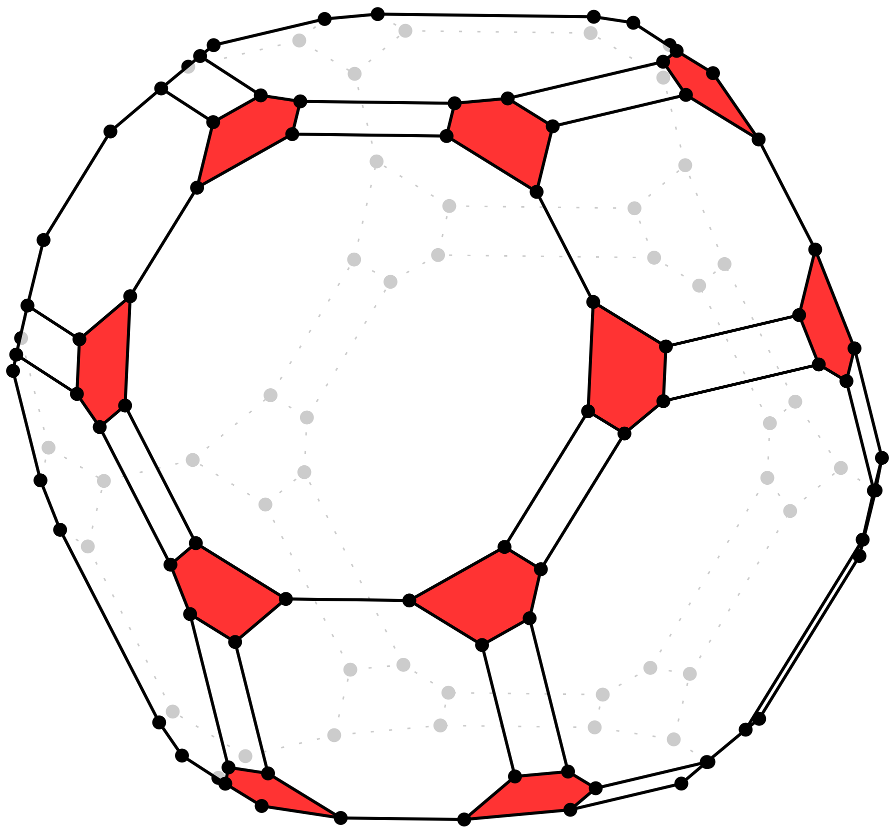
Using this together with the information we know about $\mathrm{LodayAsso}(\boldsymbol{\beta})$, we are able to construct $\mathrm{PermAsso}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ using our general strategy again.

Another Construction



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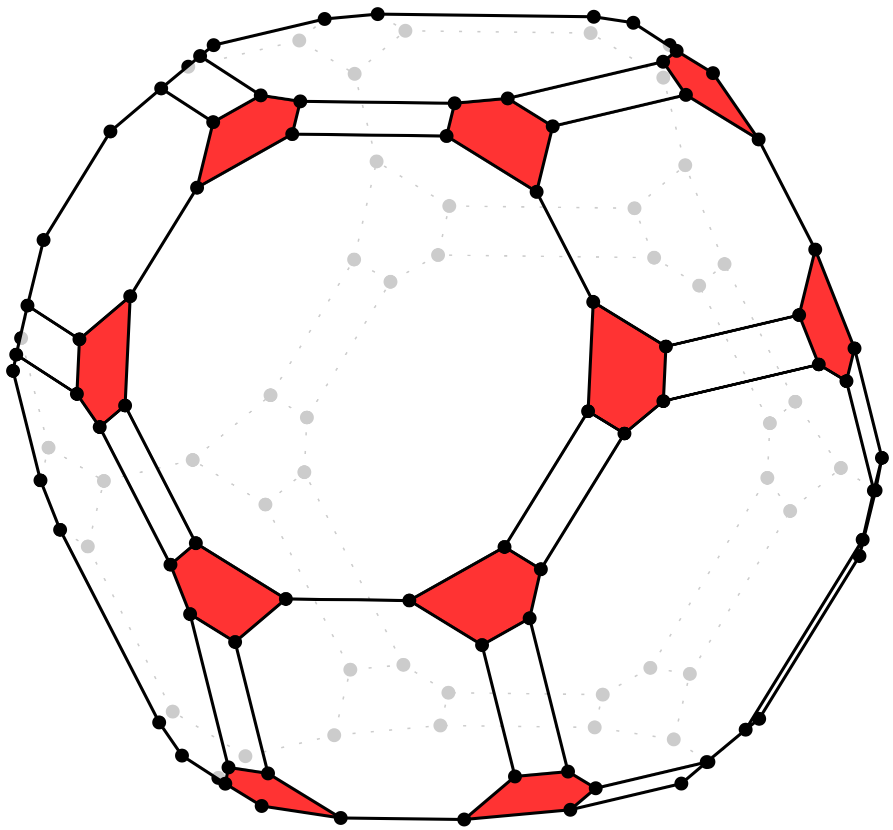

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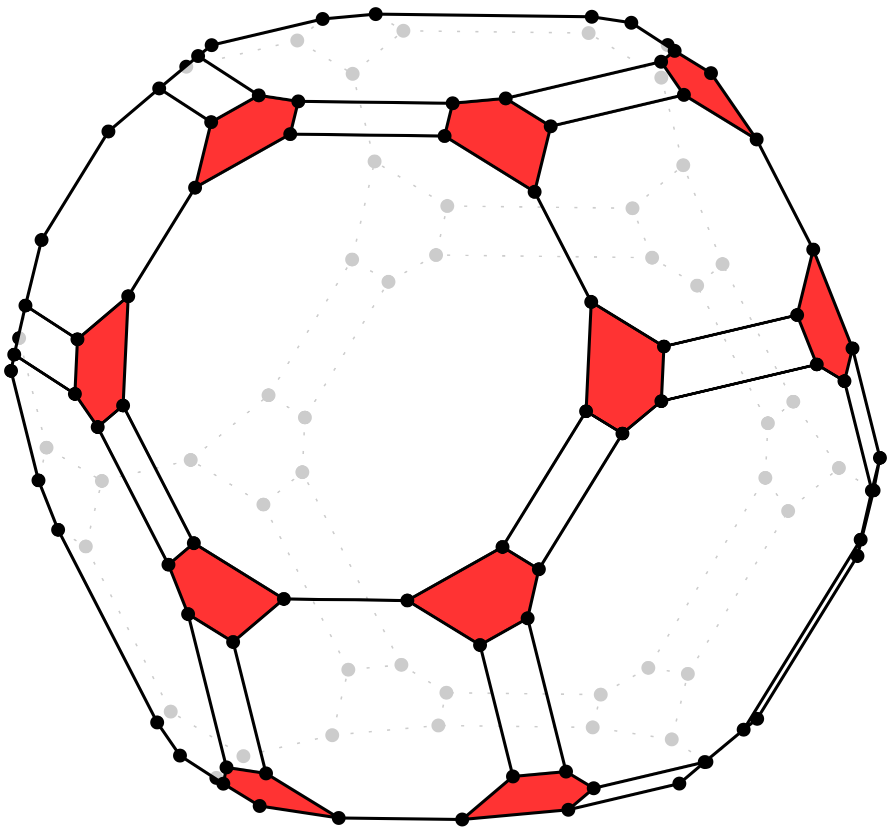

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However, it is a simple permuto-associahedron considered by Baralic-Ivanovic-Petric.

THANK YOU!