On the bond polytope

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(joint work with Markus Chimani and Alexander Nover)

(Polytop)ics: Recent advances on polytopes

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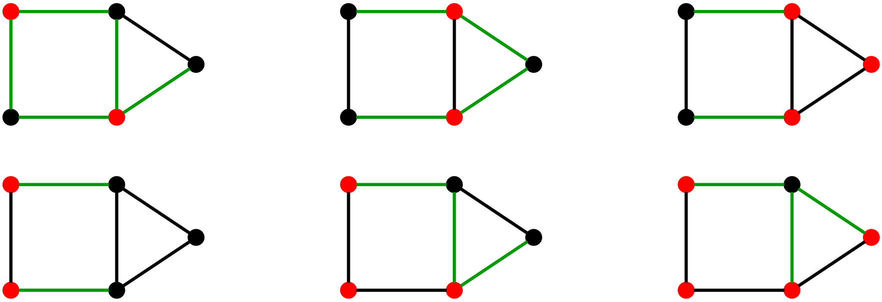
UNIVERSITÄT OSNABRÜCK

1. Introduction
2. Constructing new facets from old ones: Graph operations
3. Cycle and edge inequalities
4. $(K_5 - e)$-minor-free graphs

The main ingredient: bonds

Let $G = (V, E)$ be a graph and $S \subseteq V$.

- $\delta(S) = \{e \in E \; : \; |e \cap S| = 1\}$ is called cut.



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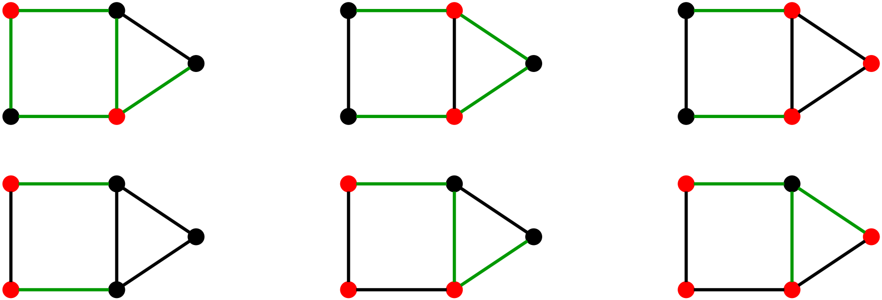
Let $G = (V, E)$ be a graph and $S \subseteq V$.

- $\delta(S) = \{e \in E \ : \ |e \cap S| = 1\}$ is called cut.
- If $G[S]$ and $G - S$ are connected, $\delta(S)$ is a bond.



The main player: bond polytopes

Let $G = (V, E)$ be a graph. For each cut δ of G we define $x_\delta \in \mathbb{R}^E$ by

$$x_\delta(e) = \begin{cases} 1, & \text{if } e \in \delta, \\ 0, & \text{otherwise.} \end{cases}$$

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The cut and bond polytope of G are defined as:

$$\mathrm{Cut}(G) = \mathrm{conv}(x_\delta : \delta \text{ is a cut in} G)$$

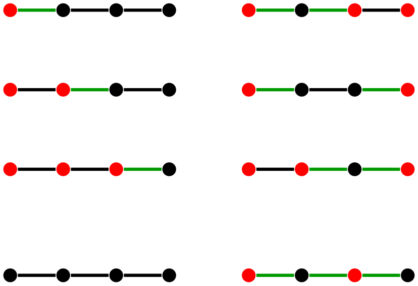
and

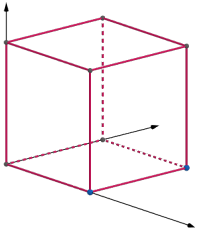
$$\mathrm{Bond}(G) = \mathrm{conv}(x_\delta : \delta \text{ is a bond in } G).$$

$\mathrm{Cut}(P_3)$ vs. $\mathrm{Bond}(P_3)$

Recall $\delta(S) = \{e \in E \ : \ |e \cap S| = 1\}$ for $S \subseteq V$

Cuts in P_3:





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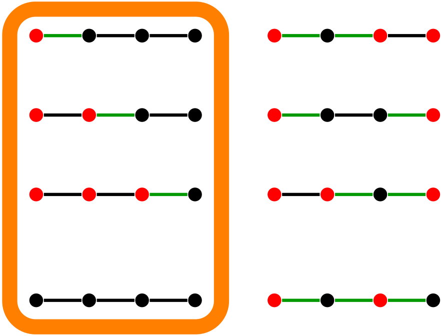
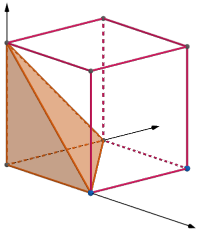


Bonds in P_3



Motivation

Max cut problem

Let $G = (V, E)$ be a graph with edge weights $(c_e)_{e \in E} \in \mathbb{R}^E$.
MaxCut: Find cut δ in G such that $\sum_{e \in \delta} c_e$ is maximal.

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- computation of ground states of Ising spin glasses,
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- network flows,
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Complexity:

- NP-complete, in general;
- polynomial time solvable for special graph classes (e.g., planar graphs).

Motivation

Max bond problem

Let $G = (V, E)$ be a graph with edge weights $(c_e)_{e \in E} \in \mathbb{R}^E$.
$\textsc{MaxBond}$: Find bond δ in G such that $\sum_{e \in \delta} c_e$ is maximal.

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- image segmentation,
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- image segmentation,
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Complexity:

- NP-complete, even on 3-connected planar or bipartite planar graphs,
- solvable in linear time on series-parallel graphs,
- (polynomial time solvable on $(K_5 - e)$-minor free graphs).

Motivation

Max bond problem

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$\textsc{MaxBond}$: Find bond δ in G such that $\sum_{e \in \delta} c_e$ is maximal.

Our starting point

- Information about $\mathrm{Cut}(G)$ and $\mathrm{Bond}(G)$ gives information about $\textsc{MaxCut}$ and $\textsc{MaxBond}$, respectively.

But: Whereas $\mathrm{Cut}(G)$ has been studied intensively, nothing is known for $\mathrm{Bond}(G)$.

How do $\mathrm{Cut}(G)$ and $\mathrm{Bond}(G)$ relate to each other?

Observation: $\mathrm{Bond}(K_n) = \mathrm{Cut}(K_n)$

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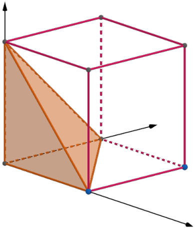
Proposition (Barahona, Mahjoub)

Let $G = (V, E)$ and δ, γ cuts in G. Then:
$\{x_\delta, x_\gamma\}$ is an edge of $\mathrm{Cut}(G)$ $\iff$ $\delta \triangle \gamma$ is a bond.



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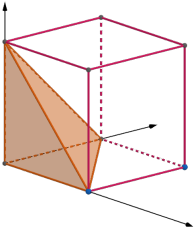
Consequently:

- Vertices of $\mathrm{Bond}(G)$ are $\mathbf{0}$ and its neighbors in $\mathrm{Cut}(G)$.
- $\mathrm{cone}(\mathrm{Bond}(G)) = \mathrm{cone}(\mathrm{Cut}(G))$
- $\dim \mathrm{Bond}(G) = |E|$



Edge deletions and contractions

Let $G = (V, E)$ and $e \in E$.

	$\mathrm{Cut}(G)$	$\mathrm{Bond}(G)$
G/e		
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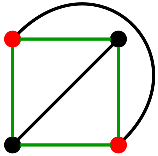
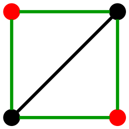
$G - e$	projection on $\{x_e = 0\}$	no nice behavior

Example

- $\delta_{K_4}(\{v, w\})$ is a bond in K_4 but $\delta_{K_4 - vw}(\{v, w\})$ is not.

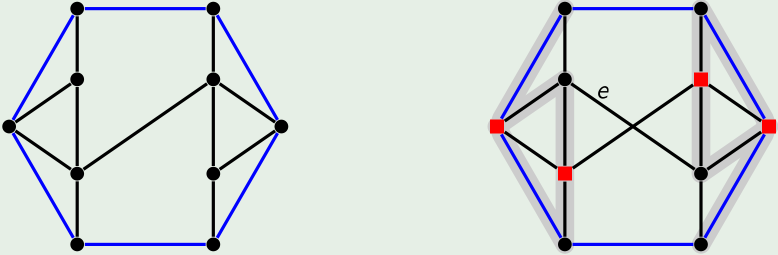




(No) facets from subgraphs and vice versa

Example

- $\sum_{e \in E(C)} x_e \leq 2$ defines a facet for $\mathrm{Bond}(G)$ but not $\mathrm{Bond}(G+e)$.



G $\qquad$ $G+e$

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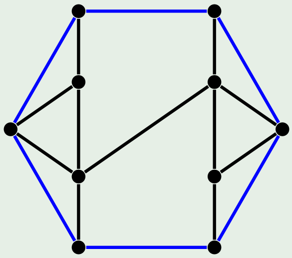
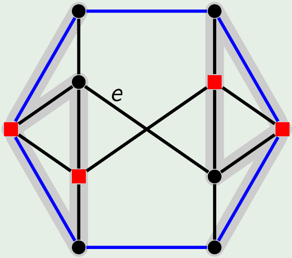


G



$G+e$

- $\sum_{e \in E(C_6)} x_e \leq 4$ defines a facet of $\mathrm{Bond}(K_{3,3})$ but not for $\mathrm{Bond}(C_6)$.

Node splitting I

Theorem

Let $G = (V, E)$, $v \in V$, *and* $a^{\mathsf{T}} x \leq b$ *be facet-defining for* $\mathrm{Bond}(G)$.

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Let $G = (V, E)$, $v \in V$, and $a^{\mathsf{T}} x \leq b$ be facet-defining for $\mathrm{Bond}(G)$.

Obtain $\overline{G} = (\overline{V}, \overline{E})$ by splitting v into v_1 and v_2.

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Theorem

Let $G = (V, E)$, $v \in V$, *and* $a^{\mathsf{T}} x \leq b$ *be facet-defining for* $\mathrm{Bond}(G)$.

Obtain $\overline{G} = (\overline{V}, \overline{E})$ *by splitting* v *into* v_1 *and* v_2.

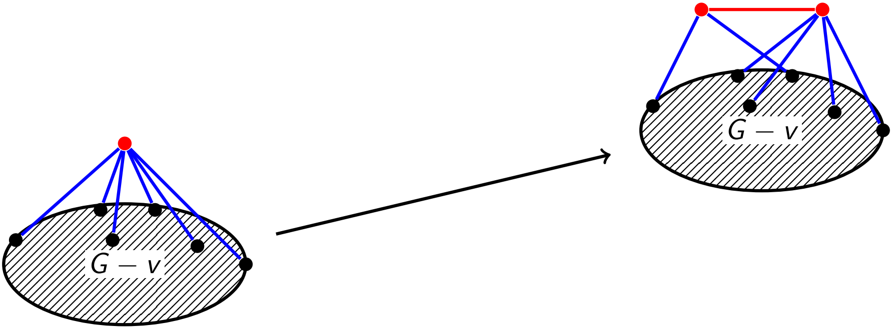
Then

$$\sum_{e \in \overline{E} \setminus \{v_1 v_2\}} a_e x_e + (b - \omega) x_{v_1 v_2} \leq b$$

defines a facet of $\mathrm{Bond}(\overline{G})$.

Here, ω *is the value of a maximum bond in* $\overline{G} - v_1 v_2$ *separating* v_1 *and* v_2 *w.r.t. edge weights induced by* a.

Node splitting II



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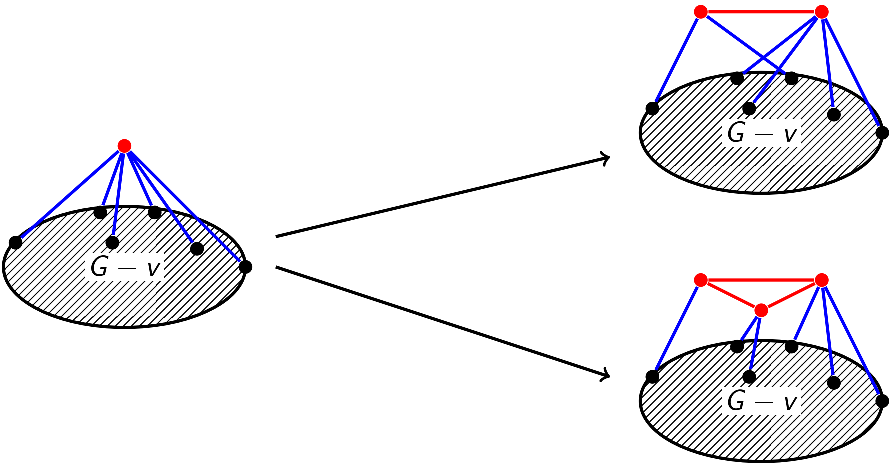
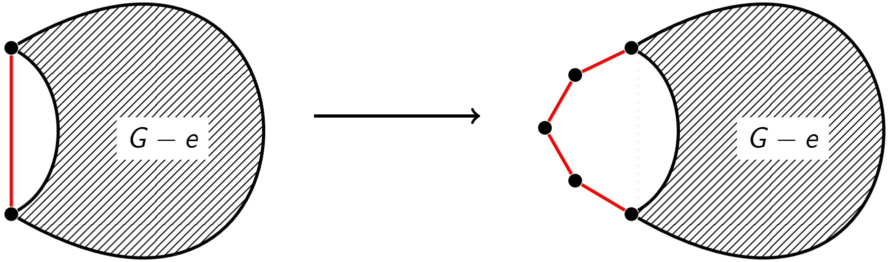


Subdividing edges and vice versa



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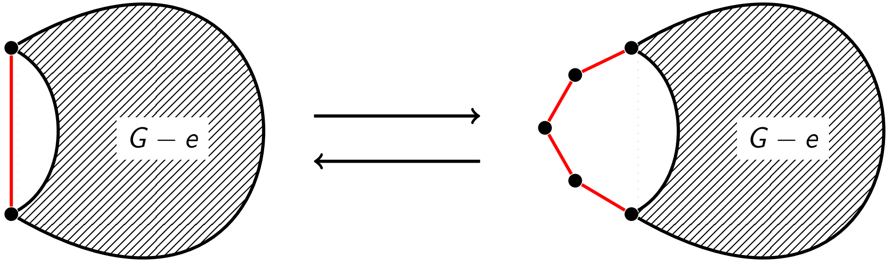
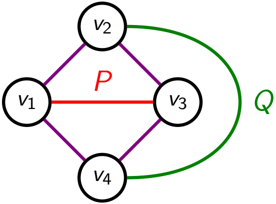


Non-interleaved cycles

A cycle $C \subseteq G$ is interleaved if there exist $v_1, v_2, v_3, v_4 \in V(C)$

- occurring along C in this order, and
- node-disjoint path P and Q in $G - E(C)$ connecting v_1 with v_3 and v_2 with v_4, respectively.

Otherwise, C is non-interleaved.



Non-interleaved cycle inequalities

Theorem

Let G be 3-connected and $C \subseteq G$ be a non-interleaved cycle. Then $\sum_{e \in E(C)} x_e \leq 2$ is facet-defining for $\mathrm{Bond}(G)$.

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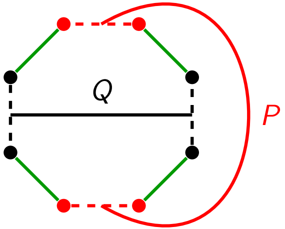
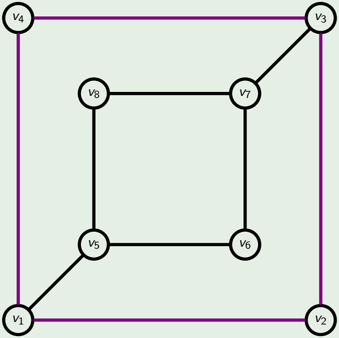
- We show that if C is a cycle, then:

$$\sum_{e \in E(C)} x_e \leq 2 \text{ is valid} \iff C \text{ is non-interleaved.}$$



2-connectedness does not suffice

Example



- The cycles v_1, v_2, v_3, v_4, v_1 and $v_1, v_2, v_3, v_7, v_6, v_5, v_1$ are non-interleaved, but do not give rise to facets.

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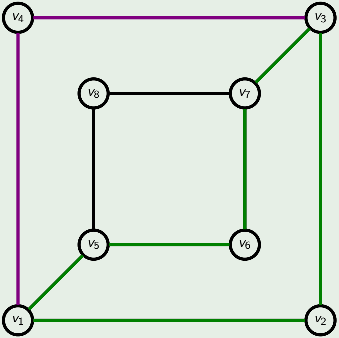

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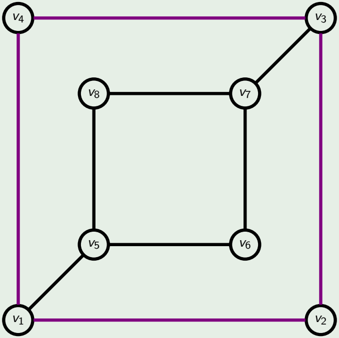

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- We provide a necessary criterion for a cycle to be facet-defining in any connected graph.

The n-cycle C_n

Theorem

The only facets of $\mathrm{Bond}(C_n)$ *are*

$$x_e \geq 0 \quad \text{for all } e \in E(C_n),$$

$$x_f - \sum_{\substack{e \in E(C_n) \\ e \neq f}} x_e \leq 0 \quad \text{for all } f \in E(C_n)\ ,$$

$$\sum_{e \in E(C_n)} x_e \leq 2.$$

Generalizations of non-interleaved cycles

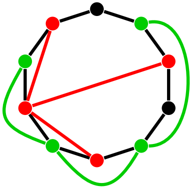
Lemma

Let $C \subseteq G$ be a cycle.

$\sum_{e \in E(C)} x_e \leq 2k$ *is valid for* $\mathrm{Bond}(G)$

$\iff$ *G does not contain a minor of the form $H = T_1 \mathbin{\dot\cup} T_2$ with*

- *T_1, T_2 disjoint trees on $k+1$ nodes,*
- *$V(T_i) \subseteq V(C)$,*
- *nodes of T_1 and T_2 alternate along C.*



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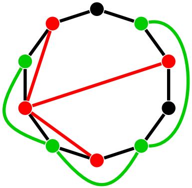
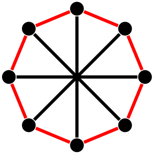


Problem

When is $\sum_{e \in E(C)} x_e \leq 2k$ facet-defining?

Examples: Wagner graphs

The generalized Wagner graph V_n ($n \in 2\mathbb{N}$) is obtained from C_n by adding the edges $\{i, i + n/2\}$ for $1 \leq i \leq n/2$.



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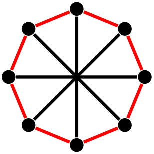
- $\sum_{e \in E(C_n)} x_e \leq 4$ is facet-defining for $\mathrm{Bond}(V_n)$.



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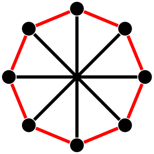
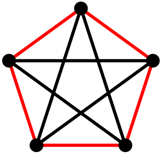


- Let $C \subseteq K_5$ be a 5-cycle. $\sum_{e \in E(C)} x_e \leq 4$ is valid but not facet-defining for $\mathrm{Bond}(K_5)$.



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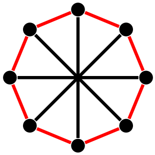

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Let $G = (V, E)$ be a connected graph, $e \in E$.

If e lies in a non-interleaved cycle, then $x_e \leq 1$ is not facet-defining.

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If e lies in a non-interleaved cycle, then $x_e \leq 1$ is not facet-defining.

Question

Is $x_e \leq 1$ facet-defining iff e does not lie in a non-interleaved cycle?

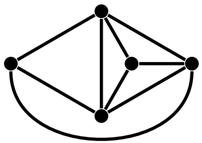
$(K_5 - e)$-minor-free graphs

Theorem

Let $G \neq K_{3,3}$ be a 3-connected $(K_5 - e)$-minor-free graph.

Then $\mathrm{Bond}(G)$ has the following facet description:

$$\begin{aligned} x_e &\geq 0 && \text{for each } e \text{ not contained in a triangle,} \\ x_e - \sum_{f \in E(C)\setminus\{e\}} x_f &\leq 0 && \text{for each induced cycle } C \text{ and } e \in E(C), \\ \sum_{e \in E(C)} x_e &\leq 2 && \text{for each non-interleaved cycle } C. \end{aligned}$$

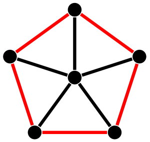


$K_5 - e$

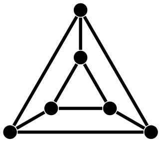
Sketch of the proof:

The only 3-connected $(K_5 - e)$-minor free graphs are

- K_3,
- $K_{3,3}$,
- *Prism*, and
- W_n $(n \geq 3)$.



W_5



Prism

□

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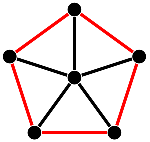
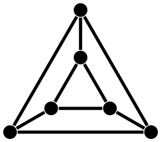
- *Prism*, and
- W_n $(n \geq 3)$.



W_5



Prism

□

Consequence:

MaxBond on $(K_5 - e)$-minor free graphs can be solved in linear time .

3-connected planar graphs

Question

Is the bond polytope of a 3-connected planar graph determined by edge and cycle inequalities?

3-connected planar graphs

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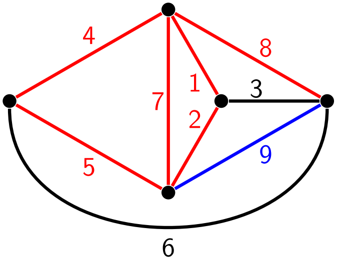
No!

$$x_1 + x_2 + x_4 + x_5 + x_7 - x_8 - x_9 \leq 2$$

defines a facet of $\mathrm{Bond}(K_5 - e)$.



Conclusion

Results:

- basic properties of $\mathrm{Bond}(G)$,
- the effect of graph operations on facets of $\mathrm{Bond}(G)$,
- interleaved cycle and edge inequalities,
- $\mathrm{Bond}(G)$ for 3-connected $(K_5 - e)$-minor-free planar graphs,
- algorithmic properties of $\textsc{MaxBond}$.

Conclusion

Results:

- basic properties of $\mathrm{Bond}(G)$,
- the effect of graph operations on facets of $\mathrm{Bond}(G)$,
- interleaved cycle and edge inequalities,
- $\mathrm{Bond}(G)$ for 3-connected $(K_5 - e)$-minor-free planar graphs,
- algorithmic properties of MaxBond.

Open problems:

- Characterize interleaved cycles that induce facets.
- When is $x_e \leq 1$ facet-defining?
- How does $\mathrm{Bond}(G)$ behave under clique sums?

Thank you!