EXERCISES ON TOPOLOGY OF REAL ALGEBRAIC VARIETIES

Exercise 1
Let C_4 be a real algebraic curve of degree 4 in $\mathbb{RP}^2$. Use Harnack's bound and Bézout's theorem to give restrictions on the possible topological types of $(\mathbb{RP}^2, \mathbb{R}C_4)$. Use Harnack method and/or Hilbert method to realise all the allowed topological types.

Exercise 2
A *nest* of a real algebraic curve is a linearly ordered (by inclusion) set of ovals of the curve. The *depth* of a nest is the number of ovals in it.

Prove that for a Harnack M-curve the depth of any its nest is at most 2.

Exercise 3
For $d \geq 1$ using the combinatorial patchworking construct

1. an empty curve of degree $2d$ in $\mathbb{RP}^2$,
2. a curve of degree $2d-1$ in $\mathbb{RP}^2$ with only one connected component,
3. a hyperbolic curve of degree d in $\mathbb{RP}^2$.

Exercise 4
Let A be a non-singular algebraic curve of degree $d = 2k$ and of type I in $\mathbb{RP}^2$. Denote by p the number of even ovals of $\mathbb{R}A$ and by n the number of odd ovals. Prove that

$$p - n \equiv k^2 \mod 4.$$

Hint: use complex orientation formula.

Exercise 5
An *hyperbolic curve* in $\mathbb{RP}^2$ is a non-singular algebraic curve A in $\mathbb{RP}^2$ such that the real part $\mathbb{R}A$ contains a nest of depth $\lfloor \frac{d}{2} \rfloor$ (a set of $\lfloor \frac{d}{2} \rfloor$ ovals any two of which form an injective pair).

1. Show that $\mathbb{R}A$ has no other ovals than those in the nest of depth $\lfloor \frac{d}{2} \rfloor$.
2. Let x be a point inside the deeper oval of the nest and let L be a real line which do not contain x. Let $\pi : \mathbb{C}A \to \mathbb{C}L$ be the restriction to $\mathbb{C}A$ of the projection on $\mathbb{C}L$ with center x. Show that $\mathbb{R}A = \pi^{-1}(\mathbb{R}L)$ and deduce that A is of type I.

Let C be a non-singular curve of degree $d = 2k$ and of type I in $\mathbb{RP}^2$.

1. Show that $\mathbb{R}C$ has at least k ovals.
2. Assume that $\mathbb{R}C$ has exactly k ovals. Show that C is hyperbolic.

Exercise 6 *Ragsdale conjecture for curves with one non empty oval*
Let A be a non-singular algebraic curve of degree $d = 2k$ in $\mathbb{RP}^2$ such that the real part $\mathbb{R}A$ contains only one non empty oval. Denote by p (resp., n) the number of even (resp., odd) ovals of $\mathbb{R}A$. Show that

$$p \leq \frac{3}{2}k(k-1) + 1 \text{ and } n \leq \frac{3}{2}k(k-1) + 1.$$

Hint: use the proof of Petrovsky's inequalities.

Exercise 7 ***1.*** Let $d \geq 0$ be a positive integer, use Bézout's theorem to show that in $\mathbb{RP}^2$ the number of connected components of a real algebraic curve of degree d is less or equal to $\frac{(d-1)(d-2)}{2} + 1$.

2. Let S be a Riemann surface of genus g equipped with an anti-holomorphic involution $\sigma : S \to S$. Show using Euler characteristic and the classification of compact surfaces that the number of connected components of $\text{fix}(\sigma)$ is less or equal to $g + 1$.

Exercise 8 *1.* Show that $\mathbb{CP}^1 \times \mathbb{CP}^1 = X$ equipped with the anti-holomorphic involution $\sigma : X \to X$ such that $\forall x = [x_0 : x_1], y = [y_0 : y_1] \in \mathbb{CP}^1$ $\sigma(x, y) = (\overline{y}, \overline{x})$ is isomorphic to a quadric ellipsoid in $\mathbb{CP}^3$.

2. Show that $\mathbb{CP}^1 \times \mathbb{CP}^1 = X$ equipped with the anti-holomorphic involution $\tau : X \to X$ such that $\forall x = [x_0 : x_1], y = [y_0 : y_1] \in \mathbb{CP}^1$ $\sigma(x, y) = (\overline{x}, \overline{y})$ is isomorphic to a quadric hyperboloid in $\mathbb{CP}^3$.

3. What is the topological classification of quadrics in $\mathbb{RP}^3 \subset \mathbb{CP}^3$?

Exercise 9
Let Q be a quadric in $\mathbb{CP}^3$ equipped with the standard complex conjugation and let A_d be a surface of degree d in $\mathbb{CP}^3$. Consider the pair $(\mathbb{R}(A_d \cap Q), \mathbb{R}Q)$. With respect to the equations of Q, describe $[\mathbb{R}(Q \cap A_d)] \in H_1(\mathbb{R}Q, \mathbb{Z})$.

Exercise 10
Show that any non-singular algebraic surface in $\mathbb{CP}^3$ is simply connected (use Lefschetz's hyperplane theorem). Deduce that the double cover of $\mathbb{CP}^2$ ramified along a non-singular curve of even degree is simply connected.

Exercise 11
Let X be a smooth projective surface over $\mathbb{C}$. We say that X is a K3-surface if $H^1(X, \mathcal{O}_X) = 0$ and the canonical bundle of X is trivial, i.e. $K_X = \mathcal{O}_X$ Show that a complex surface $X \subset \mathbb{CP}^n$ is a K3-surface in the following cases:

1. X is a smooth, degree-4 hypersurface in $\mathbb{CP}^3$.
2. X is the smooth complete intersection of a cubic and a quadric $\mathbb{CP}^4$
3. X is the smooth complete intersection of three quartics $\mathbb{CP}^5$.

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References

[1] Viro, O. Ya., *Patchworking real algebraic varieties*, preprint, 1995. Available at `www.pdmi.ras.ru/~olegviro/pw.ps`.