EXERCISES ON CONVEX ALGEBRAIC GEOMETRY

Exercise 1
Give an example of a convex semi-algebraic body in $\mathbb{R}^3$ with zero-dimensional faces with each of one, two, and three-dimensional normal cones.

Exercise 2
Consider the convex cone of univariate polynomials of degree ≤ 4 that are nonnegative on the interval $[-1, 1]$:

$$P \ = \ \{(a, b, c, d, e) \in \mathbb{R}^5 \text{ such that } at^4 + bt^3 + ct^2 + dt + e \geq 0 \text{ for all } t \in [-1, 1]\}.$$

- What is the algebraic boundary of P? What are its extreme rays?
- What are the faces and algebraic boundary of the convex hull of

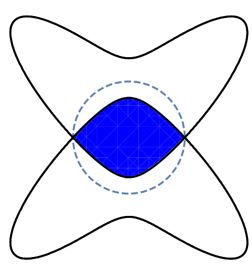
$$C = \big(\{(t, t^2, t^3, t^4) \text{ such that } t \in [-1, 1]\}\big)?$$

- How is $\operatorname{conv}(C)$ related to P?
- Are either of P or $\operatorname{conv}(C)$ a spectrahedron? Or the projection of one?

Exercise 3
Consider the convex semi-algebraic set

$$C \ = \ \{(x, y) \in \mathbb{R}^2 : f(x, y) \geq 0 \ \text{ and } \ 2 \geq x^2 + y^2\}$$

where $f(x, y) = x^4 - x^2y^2 - 4x^2 + y^4 - 5y^2 + 4$.



- Calculate the algebraic boundary of the convex dual of C.
- For $c = (3, 1)$ and $c = (1, 3)$, calculate the minimal polynomial over $\mathbb{Q}$ of

$$c^* \ = \ \max c_1x + c_2y \ \text{ such that } (x, y) \in C.$$

- For each, write the optimal point (x, y) using the field extension $\mathbb{Q}(c^*)$. That is, find polynomials $p, q, r \in \mathbb{Q}[t]$ for which the optimal point is $\left(\frac{p(c^*)}{r(c^*)}, \frac{q(c^*)}{r(c^*)}\right)$.

Exercise 4
What can we say about the convex hull of the curve $F_{2k} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^{2k} + x_2^{2k} = 1\}$,

for $k \geq 3$? Can you construct an explicit semidefinite lift for the convex hull of F_{2^k}, $k \geq 2$, whose sizeis linear in k? And what about the limit $Q := \lim_{k \mapsto \infty} F_{2k}$?

Exercise 5

Is a stadium (the convex hull of the union of two circles of the same radius in $\mathbb{R}^2$) a spectrahedron and/or a spectrahedral shadow? What if one allows for two different radius $R > r$?

Exercise 6

Show that the unit ball in $\mathbb{R}^n$ with respect to the Euclidean norm is a spectrahedron. What is the smallest possible size of matrices in the spectrahedral description for $n = 1, 2, 3$?

Exercise 7

Show that a hyperbolicity cone is a basic closed semi-algebraic set.

Exercise 8

Let A be an $\mathbb{R}$-algebra which is finite dimensional as an $\mathbb{R}$-vector space. Every element $a \in A$ defines an endomorphism $m_a : A \to A$, $x \mapsto a \cdot x$. We write $\mathrm{tr}_{A/\mathbb{R}}(a) := \mathrm{tr}(m_a)$. Furthermore, we consider the symmetric bilinear form $B : A \times A \to \mathbb{R}$, $(a, b) \mapsto \mathrm{tr}_{A/\mathbb{R}}(a \cdot b)$. Show the following:

a) Let A be a local ring with maximal ideal $\mathfrak{m}$. What is the rank and the signature of B when $A/\mathfrak{m} \cong \mathbb{R}$ resp. $A/\mathfrak{m} \cong \mathbb{C}$?
b) Now let $A = \mathbb{R}[x_1, \ldots, x_n]/I$ where I is the ideal generated by polynomials $f_1, \ldots, f_r$ which have only finitely many common complex zeros. Then the number of complex zeros of $f_1, \ldots, f_r$ is the rank of B and the number of real zeros is the signature of B.

Exercise 9

Let X be a noetherian integral separated regular scheme of dimension one with function field K. The *narrow class group* $\mathrm{Cl}^+(X)$ is the group of Weil divisors divided by the subgroup of all principal divisors of the form $(g_1^2 + \ldots + g_r^2)$ with $g_i \in K^*$.

a) The kernel of the natural homomorphism $\mathrm{Cl}^+(X) \to \mathrm{Cl}(X)$ is a 2-torsion group.
b) Compute $\mathrm{Cl}^+(X)$ when $X = \mathrm{Spec}(\mathbb{Z}[\sqrt{n}])$ for $n = 1, 2, 3$.
c) Let X be a projective curve over $\mathbb{R}$ and let $D \in \mathrm{Cl}^+(X)$. Show that $D = 2E$ for some $E \in \mathrm{Cl}^+(X)$ if and only if the multiplicity of D is even in every real point of X.

Exercise 10

(1) Let $M \subset \mathbb{R}^n$ be a spectrahedral shadow, show that M^* is a spectrahedral shadow.
(2) Show that the dual convex cone Σ_{2d}^* of the cone of sums of squares of homogeneous polynomials of degree d is a spectrahedron.
(3) Show that the cone $\Sigma_{2d} \subset \mathbb{R}[x_1, \cdots, x_n]_{2d}$ of sums of squares of homogeneous polynomials of degree d is a spectrahedral shadow, but not a spectrahedron if $n, d > 1$.

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