LEONHARD EULER (1707–1783)

EULER AND THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY

Read Euler, he is the master of us all.

Marquis de Pierre Simon Laplace (1749–1827)

The Euler "Calculus of Variations" of the year 1744 is one of the most beautiful mathematical works that has ever been written.

Constantin Carathéodory (1873–1950)

Mathematics knows, besides the exclusive era of the Greeks, no luckier constellation than the one under which Euler was born. It was up to him to give mathematics a completely changed form and to shape it into the powerful edifice that it is today.

Andreas Speiser (1885–1970)

Seen statistically, Euler must have made a discovery every week . . .
Euler was not only one of the greatest mathematicians, but also in general one of the most creative human beings.

Rüdiger Thiele

Euler did not sour his life with limiting value considerations, convergence and continuity criteria and he could not and did not wish to bother about the logical foundation of analysis, but rather he relied – only on occasion unsuccessfully – on his astonishing certitude of instinct and algorithmic power.

Emil Alfred Fellmann

FUNDAMENTAL FORCES

(i) gravitation

(ii) elecromagnetic interaction

(iii) strong interaction (nuclear forces)

(iv) weak interaction (e.g., radioactive decay)

DARK MATTER

Universe:

classical matter and energy: 4%

dark matter and dark energy: 96%

CHALLENGE FOR THE 21TH CENTURY

(i) Create a rigorous unified theory for the fundamental forces in the universe (quantum gravity)

(ii) Understand the structure of dark matter and dark energy

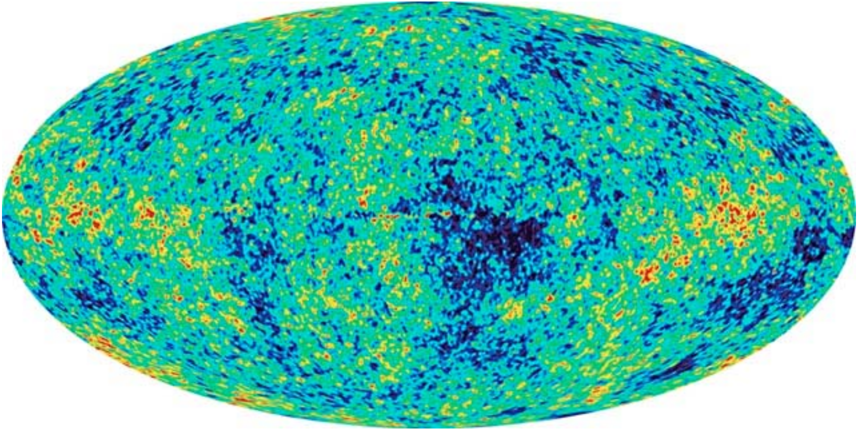
STATE OF THE ART

(a) Standard Model in elementary particle physics

CERN 2009 (LARGE HADRON COLLIDER):
Higgs particle, supersymmetric particles

(b) Standard Model in cosmology

age of the universe: $13.7 \cdot 10^9$ years
NASA 2007 (WMAP data look at 370 000 years
after the Big Bang)



GENERAL PRINCIPLES

(i) **The infinitesimal strategy**
differential equations

(ii) **Optimality** (principle of least action; Euler!)
local symmetry
curvature of fiber bundles

$$\boxed{FORCE = CURVATURE}$$

cohomology (characteristic classes)

(iii) **Quantization**
principle of least action plus
stochastic quantum fluctuations
quantum numbers
representations of groups
topological invariants
deformation of mathematical structures
(e.g., Poisson structures)

GENERAL PRINCIPLES (continued)

(iv) **Renormalization**
$\lim_{N\to\infty} \ldots$ (N degrees of freedom)
extract finite information from infinities (Euler!)
transform infinities into effective physical
quantities (e.g., mass, charge)

(v) **Symmetry**
Groups
Hopf algebras ("dual groups")
supersymmetry: CERN 2009
(add Grassmann variables, $xy = -yx$,
graded mathematical structures)
motivic Galois group (abstract renormalization group)

GENERAL PRINCIPLES (continued)

(vi) **Many-particle systems**

universal principle: *generating function*

partition function

Feynman functional integral

(correlation functions)

Euler-Riemann zeta function

$$\boxed{\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s > 1}$$

$$Z = \sum_{n=1}^{N} \mathrm{e}^{-E_n/kT}$$

$$\int e^{\mathrm{i}S[q]/\hbar} \mathcal{D}q$$

TASK of MATHEMATICS

Decode information !!!

Solve differential equations

Construct correlation functions from

Feynman functional integrals

Prove the Riemann conjecture!

EVOLUTION LAW IN MATHEMATICS

(i) numbers and geometric objects

(ii) mathematical structures (e.g., groups)

(iii) *functors* between categories
of mathematical structures
(e.g., homology; Euler!)

(iv) *statistics of mathematical structures*
via Feynman functional integrals (Witten!)

THE PRINCIPLE OF LEAST ACTION

$$\boxed{S(\psi) = \min!}$$

Bees – by virtue of a certain geometrical forethought – know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material.

Pappos of Alexandria, 300 BC

Every process in nature will occur in the shortest possible way.

Leonardo da Vinci (1452–1519)

A light ray between two points needs the shortest possible time.

Pierre de Fermat (1601–1665)

THE PRINCIPLE OF LEAST ACTION

$$\boxed{S(\psi) = \min!}$$

Johann Bernoulli, professor of mathematics, greets the most sophisticated mathematicians in the world. Experience shows that noble intellectuals are driven to work for the pursuit of the knowledge by nothing more than being confronted with difficult and useful problems.
Six month ago, in the June edition of the Leipzig Acta Eruditorum (journal of scientists), I presented such a problem. The allotted six-month deadline has now gone by, but no trace of a solution has appeared. Only the famous Leibniz informed me that he had unravelled the knot of this brilliant and outstanding problem, and he kindly asked me to extend the deadline until next Easter. I agreed to this honorable request...

Johann Bernoulli, January 1697

THE PRINCIPLE OF LEAST ACTION

$$\boxed{S(\psi) = \mathrm{min!}}$$

$$S(\psi) = \textit{action} = \textit{energy} \times \textit{time}$$

Since the divine plan is the most perfect thing there is, there can be no doubt that all actions in the universe can be determined by the calculus of the minima and maxima from the corresponding causes.

Leonard Euler (1707-1783)

THE PRINCIPLE OF LEAST ACTION

$$\boxed{S(\psi) = \min!}$$

The history of the principle of least action has often been described. Yet the matter is still controversial, and there seems to be no general agreement who invented the principle, Leibniz (1646–1717), Euler (1707–1783), or Maupertuis (1698–1759)...

We mention that the first mathematical treatment of the action principle was given by Euler in the Additamentum of his Methodus invendiendi (A method for finding curves which have a minimal or maximal property or solutions of the generalized isoperimetric problem),
Bousquet, Lausannae et Genevae 1744.

Mariano Giaquinta and Stefan Hildebrandt, 1996

THE PRINCIPLE OF LEAST ACTION

$$\boxed{S(\psi) = \min!}$$

By generalizing the method of Euler in the calculus of variations, Lagrange (1736–1813) discovered, how one can write, in a single line, the basic equation for all problems in analytic mechanics.

Carl Gustav Jakob Jacobi (1804–1851)

But since the rules for solving the isoperimetric problem were not of sufficient generality, the famous Euler undertook to refer all investigations of this type to a general method.
But even as sophisticated and fruitful as his method is, one must nevertheless confess that it is not sufficiently simple... Now here one finds a method which requires only a simple use of the principles of differential and integral calculus.

Joseph Louis Lagrange, 1762

As I see, your analytic solution of the isoperimetric problem contains all that one can wish for in this situation. I am very happy that this theory which I have treated since the first attempts almost alone, has been brought precisely by you to the highest degree of perfection.

The importance of the situation has occasioned me with the help of your new insights to myself conceive of an analytic solution, but which I shall not make known before you have published your deliberations, in order not to deprive you of the least part of the fame due you.

Euler, in a letter to the young Lagrange

THE PRINCIPLE OF LEAST (CRITICAL) ACTION IN COSMOLOGY

$$\int_{\mathcal{M}_4} R \; \mathrm{d}\mathit{vol} = \mathit{critical}!$$

$$\boxed{\int_{\mathcal{M}_4} (R - T) \; \mathrm{d}\mathit{vol} = \mathit{critical}!}$$

THE PRINCIPLE OF LEAST (CRITICAL) ACTION IN PARTICLE PHYSICS

$$\boxed{\int_Q L \, \mathrm{d}x\mathrm{d}t = \textit{critical}!}$$

$$L := \psi^\dagger \psi_t + \psi^\dagger \psi_x$$

(i) Global symmetry:

$$\psi_+(x,t) = \mathrm{e}^{\mathrm{i}\alpha}\psi(x,t)$$

(ii) Local symmetry:

$$\alpha = \alpha(x,t)$$

Change Lagrangian:

$$\boxed{L = \psi^\dagger \nabla_t \psi + \psi^\dagger \nabla_x \psi}$$

$$\nabla_t = \frac{\partial}{\partial t} + U(x,t), \qquad \nabla_x = \frac{\partial}{\partial x} + A(x,t).$$

Gauge force (curvature):

$$F := \nabla_x \nabla_t - \nabla_t \nabla_x = U_x - A_t.$$

$$\boxed{\mathcal{F} = D\mathcal{A}}$$

Force = Curvature

THE PRINCIPLE OF LEAST (CRITICAL) ACTION IN PARTICLE PHYSICS

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + U\psi$$

$$\psi(x,t) = \int_{\mathbb{R}} \mathcal{K}(x,t;x_0,t_0)\psi(x_0,t_0)dx_0.$$

$$\boxed{\mathcal{K}(x,t;x_0,t_0) = \int \mathrm{e}^{\mathrm{i}S[q]/\hbar}\,\mathcal{D}q}$$

$q(t_0) = x_0, \quad q(t) = x$

Explicitly:

$$\int \mathrm{e}^{\mathrm{i}S[q]/\hbar}\mathcal{D}q = \lim_{N\to\infty} \frac{1}{l^{N+1}} \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}S_N[q]/\hbar} dq_1 \cdots dq_N$$

One frequently encounters formulas due to Euler!
(e.g., gamma function, zeta function, beta function)

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\gamma\langle x|Ax\rangle} \frac{dx_1}{\sqrt{2\pi}} \cdots \frac{dx_N}{\sqrt{2\pi}} = \frac{\mathrm{e}^{-\frac{1}{2}\zeta_A(0)\ln\gamma}}{\sqrt{\det A}}$$

$$\zeta_A(s) = \sum_{n=1}^{N} \frac{1}{\lambda_n^s}$$

$$\det A = \lambda_1 \cdots \lambda_N = \mathrm{e}^{-\zeta_A'(0)}.$$

$$N \to \infty$$

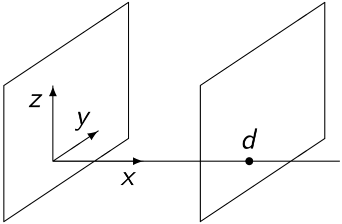


Figure: The Casimir effect

Ground state energy of the photon field:

$$E(d) = \frac{\pi c\hbar}{8L} \sum_{l,m,n\in\mathbb{Z}} (l^2 + m^2 + \lambda^2 n^2)^{1/2} \quad , \quad \lambda := L/d.$$

$$Z(\lambda, s) := {\sum_{l,m,n\in\mathbb{Z}}}' \frac{1}{(m^2 + n^2 + \lambda^2 n^2)^s} \quad , \quad \Re(s) > 3/2.$$

THE CASIMIR EFFECT AND RENORMALIZATION (continued)

physics textbooks: Euler–MacLaurin summation formula

rigorous approach: analytic continuation: $s = -1/2$

Don Zagier: asymptotics for small d

Force between the two plates:

$$\boxed{F = -E'(d) = -\frac{\pi^2 \hbar c L^2}{240 d^4}}$$

Nature sees analytic continuation!

E. Z.

Quantum Field Theory:
A Bridge between Mathematicians and Physicists

I: *Basics in Mathematics and Physics*, Springer, 2006

II: *Quantum Electrodynamics,* 2008
The Importance of Light:
basis of both our biological existence and our scientific knowledge

III: *Gauge Theory*
Force $=$ Curvature

IV: *Quantum Mathematics*
Methods of Quantization

V: *The Physics of the Standard Model*

VI: *Quantum Gravity and String Theory*

B. Fauser, J. Tolksdorf, E.Z. (eds.)
Quantum Gravity, Birkhäuser, 2006

A. Connes, M. Marcolli, Non-Commutative Geometry and the Standard Model in Particle Physics
(to appear 2007)

COUNTERTERMS/RENORMALIZATION

(i) Rational function: pole at $z = \mathrm{i}$

$$p(z) = \frac{a_1}{z - \mathrm{i}} + \ldots + \frac{a_n}{(z - \mathrm{i})^n}$$

(ii) Meromorphic function: poles at $z_1, z_2, \ldots$

$$f(z) = \sum_{k=1}^{\infty} p_k(z)$$

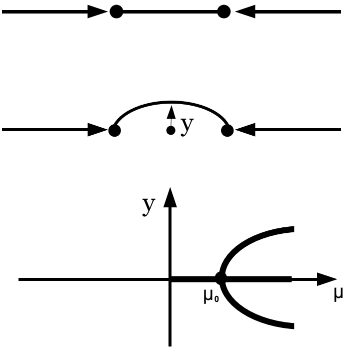
counterterms (Mittag–Leffler theorem):

$$f(z) = \sum_{k=1}^{\infty} (p_k(z) - c_k(z))$$

BPHZ renormalization

combinatorics of counterterms: Hopf algebras

BIFURCATION



EPILOGUE

Mathematics is an organ of knowledge and an infinite refinement of language.

It grows from the usual world of intuitions as does a plant from the soil, and its roots are the numbers and simple geometric intuitions.

We cannot imagine into what depths and distances this spiritual eye will lead us.

Erich Kähler (1906–2000)

By a particular prerogative, not only does each man advance day by day in the sciences, but all men together make continual progress as the universe ages ...

Thus the entire body of mankind as a whole, over many centuries, must be considered as a single man, who lives forever and continues to learn.

Blaise Pascal (1623–1662)

He was a great scholar and a gracious human being.

Inscription on the Euler memorial tablet
in Riehen (Basel)