Epilogue

Mathematics is the gate and the key to the sciences.
Roger Bacon (1214–1294)

I love mathematics not only because it is applicable to technology but also because it is beautiful.
Rózsa Péter (1905–1977)

The perfection of mathematical beauty is such whatsoever is most beautiful is also found to be most useful and excellent.
D’Arcy Wentworth Thompson (1860–1948)

The observation which comes closest to an explanation for the mathematical concepts cropping up in physics which I know is Einstein’s statement that the only physical theories we are willing to accept are the beautiful ones.
Eugene Wigner (1902–1995)

A truly realistic mathematics should be conceived, in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautious attitude toward hypothetic extensions of its foundations as is exhibited by physics.
Hermann Weyl (1885–1955)

The interplay between generality and individuality, deduction and construction, logic and imagination – this is the profound essence of live mathematics.

Any one or another of the aspects can be at the center of a given achievement. In a far-reaching development all of them will be involved. Generally speaking, such a development will start from the “concrete ground,” then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observations are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual “reality.”

In brief, the flight into abstract generality must start from and return to the concrete and specific.\(^\text{13}\)

Richard Courant (1888–1972)

\(^{13}\) Mathematics in the modern world, Scientific American 211(3) (1964), 41–49 (reprinted with permission).
There are mathematicians who reject a binding of mathematics to physics, and who justify mathematical work solely by aesthetical satisfaction which, besides all the difficulty of the material, mathematics is able to offer. Such mathematicians are more likely to regard mathematics as a form of art than science, and this point of view of mathematical unselfishness can be characterized by the slogan “l’art pour l’art”.

On the other hand, there are physicists who regret that their science is so much related to mathematics. They fear a loss of intuition in the natural sciences. They consider the intimate relation with nature, the finding of ideas in nature itself, which was given to Goethe (1749–1832) in such a high degree, as being destroyed by mathematics, and their anger or sorrow is the more serious the more they are forced to realize the inevitability of mathematics.

Both points of view deserve serious consideration; because not only people with narrow minds have expressed such opinions. Yes, one can say that such a radical inclination to one side or the other, if not caused by a lack of talent, is sometimes evidence of a deeper perception of science, as if someone is interested in both sciences, but at the same time is satisfied with obvious connections between mathematics and physics...

Mathematics is an organ of knowledge and an infinite refinement of language. It grows from the usual language and world of intuitions as does a plant from the soil, and its roots are the numbers and simple geometrical intuitions. We do not know which kind of content mathematics (as the only adequate language) requires; we cannot imagine into what depths and distances this spiritual eye will lead us.14

Erich Kähler (1906–2000)

The most vitally characteristic fact about mathematics, in my opinion, is its quite peculiar relationship to the natural sciences, or more generally, to any science which interprets experience on a higher than purely descriptive level...

I think that this is a relatively good approximation to truth – which is much too complicated to allow anything but approximations – that mathematical ideas originate in empirical facts, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetic motivations, than to anything else and, in particular, to an empirical science.

But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant tributaries, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical sources or after much abstract inbreeding, a mathematical object is in danger of degeneration. At the inception, the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up...

Whenever this stage is reached, then the only remedy seems to be a rejuvenating return to the source: the re-injection of more or less directly empirical ideas. I am convinced that this is a necessary condition to con-

14 On the relations of mathematics to physics and astronomy (in German), Jahresberichte der Deutschen Mathematiker-Vereinigung 51 (1941), 52–63 (reprinted with permission).
serve the freshness and the vitality of the subject and that this will remain
equally true in the future.$^{15}$

John von Neumann (1903–1957)

I want to say a word about the communication between mathematicians
and physicists.
It has been very bad in the past, and some of the blame is doubtless to be
laid on the physicist’s shoulders. We tend to be very vague, and we don’t
know what the problem is until we have already seen how to solve it. We
drive mathematicians crazy when we try to explain what our problems are.
When we write articles we don’t do a good enough job of specifying how
certain we are about our statements; we do not distinguish guesses from
theorems.
On the other hand, since I have said a lot of nice things about mathematics,
I have to say that the mathematicians carry an even greater burden of guilt
for this communication problem, largely because of their elitism. They
often have, it seems to me, as their ideal the savant who is understandable
only to a few co-specialists and who writes articles that one has to spend
years to try to fathom.
When physicists write articles, they generally start them with a paragraph
saying, “Up until now, this has been thought to be the case. Now, so –
and – so has pointed out this problem. In this article, we are going to
try to suggest a resolution of this difficulty.” On the other hand, I have
seen books of mathematics, not just articles but books, in which the first
sentence in the preface was, “Let $H$ be a nilpotent subgroup of . . . ” These
books are written in what I would call a lapidary style. The idea seems
to be that there should be no word in the book that is not absolutely
necessary, that is inserted merely to help the reader to understand what
is going on.
I think this is getting much better. I find it is wonderful how mathemati-
cians these days are willing to explain their field to interested physicists.
This situation is improving, partly because as Iz Singer mentioned, we re-
alize now that in certain areas we have much more in common than we
had thought, but I think a lot more has to be done. There is still too much
mathematics written which is not only not understandable to experimental
or theoretical physicists, but is not even understandable to mathematicians
who are not the graduate students of the author.$^{16}$

Steven Weinberg (born 1933)

Relations between mathematics and physics vary with time. Right now,
and for the past few years, harmony reigns and a honeymoon blossoms.
However, I have seen other times, times of divorce and bitter battles, when
the sister sciences declared each other as useless — or worse. The following
exchange between a famous theoretical physicist and an equally famous
mathematician might have been typical, some fifteen or twenty years ago:
Says the physicist: “I have no use for mathematics. All the mathematics I
ever need, I invent in one week.”

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R. Heywood, University of Chicago Press, 1947 (reprinted with permission).

$^{16}$ Mathematics: The unifying thread in science: Notices Amer. Math. Soc. 33
(1986), 716–733 (reprinted with permission).
Answers the mathematician: “You must mean the seven days it took the Lord to create the world.”

A slightly more reliable document is found in the preface of the first edition of Hermann Weyl’s book on group theory and quantum mechanics from 1928. He writes: “I cannot abstain from playing the role of an (often unwelcome) intermediary in this drama between mathematics and physics, which fertilize each other in the dark, and deny and misconstrue one another when face to face.”

This dramatic situation, described here by one of the great masters in both sciences, is a result of recent times. At the time of Newton (1643–1727) disharmony between mathematics and physics seemed unthinkable and unnatural, since both were his brainchildren; and close symbiosis persisted through the whole of the eighteenth century. The rift arose around 1800 and was caused by the development of pure mathematics (represented by number theory) on the one hand, and of a new kind of physics, independent of mathematics, which developed out of chemistry, electricity and magnetism on the other. This rift was widened in Germany under the influence of Goethe (1749–1832) and his followers, Schelling (1775–1854) and Hegel (1770–1831) and their “Naturphilosophie”.

Our protagonists are Carl Friedrich Gauss (1777–1855), as the creator of modern number theory, and Michael Faraday (1791-1867) as the inventor of physics without mathematics (in the strict sense of the word).

It would be foolish, of course, to claim the nonexistence of number theory before Gauss. An amusing document may illustrate the historical development. Erich Hecke’s famous Lectures on the Theory of Algebraic Numbers has on its last page a “timetable”, which chronologically lists the names and dates of the great number theoreticians, starting with Euclid (300 B.C.) and ending with Hermann Minkowski (1864–1909). As a physicist, I am impressed to find so many familiar names in this Hall of Fame: Fermat (1601–1665), Euler (1707–1783), Lagrange (1736–1813), Legendre (1752–1833), Fourier (1768–1830), and Gauss. In fact, we cannot find a single great number theoretician before Gauss, whom we would not count among the great physicists, provided we disregard antiquity. Specialization starts after 1800 with names like Kummer (1810–1891), Galois (1811–1832), and Eisenstein (1823–1852); who were all under the great influence of Gauss’ Disquisitiones arithmeticae from 1801. In this specific sense, Gauss’ book marks the dividing line between mathematics as a universal science and mathematics as a union of special disciplines, and between the “géomètre” as a universal “savant” in the sense of the eighteenth century and the specialized “mathématicien” of modern times. As is typical for a man of transition, Gauss does not belong to either category, he was universal and specialized. The struggle raged within him – and made him suffer.

Res Jost (1918–1990)

Mathematics and physics since 1800: discord and sympathy

By a particular prerogative, not only does each man advance day by day in the sciences, but all men together make continual progress as the universe ages... Thus, the entire body of mankind as a whole, over many centuries, must be considered as a single man, who lives forever and continues to learn.

Blaise Pascal (1623–1662)

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