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The length of the shadow boundary

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Abstract

Among all embedded manifolds with positive exterior curvature $\leq k$ the ratio between the length of the shadow boundary and the surface is maximized by the sphere of radius $1/k$.

Let $\tilde{M}$ be a d–dimensional closed oriented manifold. We consider an embedding

$$I : \tilde{M} \to \mathbb{R}^{d+1}, \qquad M := I(\tilde{M})$$

in Euclidean $(\mathbb{R}^{d+1}, \bar{g})$ with the outward unit normal vector field $N : M \to T_M\mathbb{R}^{d+1}$.

$V(M)$ denotes the volume of (M, g) w.r.t. the Riemannian metric $g := \bar{g}\restriction_{TM}$ or, equivalently, w.r.t. the d–form

$$\omega := \mathbf{i}_N \bar{\omega}\restriction_{T_M\mathbb{R}^{d+1}}$$

on M. Here $\bar{\omega} := dx \wedge \hat{\omega}$ with $\hat{\omega} := dx_1 \wedge \ldots \wedge dx_d$ is the Euclidean volume on $\mathbb{R}^{d+1} \equiv \mathbb{R} \times \mathbb{R}^d$.

The *shadow* of the hypersurface M under the *vertical* orthogonal projection $P : \mathbb{R}^{d+1} \equiv \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is denoted by

$$\hat{S} := P(M) \subset \mathbb{R}^d,$$

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and we are interested in the $d-1$–dimensional Hausdorff measure $L(M)$ of its topological boundary

$$\hat{B} := \partial \hat{S} \subset \mathbb{R}^d,$$

the *length of the shadow boundary.*

Fixing the volume $v = V(M)$ does not lead to an upper bound of $L(M)$:

- For $d = 1$ the manifold is a union $\tilde{M} = S^1 \cup \ldots \cup S^1$ of n circles, so that we have $L(M) = 2n$ if the shadow intervals are disjoint, whereas we have $V(M) = v$ for circles of radius $v/(2\pi n)$.
- For $d \geq 2$ and arbitrary $l > 0$ there is a linear transformation f on $\mathbb{R}^{d+1}$ with $V(f(M)) = V(M)$ and $L(f(M)) = l$.

However, if we in addition fix the maximal exterior curvature

$$k_{\max}(M) := \max_{m \in M} \|A_m\| > 0$$

of the embedding, $A_m : T_m M \to T_m M$ being the self-adjoint Peterson operator of the second fundamental form, then the length of the shadow boundary is bounded above:

Theorem 1 *For $k_{\max}(M) = k$*

$$L(M) \geq L(S_k^d), \tag{1}$$

and if A is positive definite,

$$L(M) \leq \frac{V(M)}{V(S_k^d)} \cdot L(S_k^d), \tag{2}$$

S_k^d being the d–sphere of radius $1/k$.

Remarks 2

- *The assumption $A > 0$ is rather restrictive. Indeed one may weaken it by assuming that each orbit of the gradient flow (see below) meets $P^{-1}(\hat{B}) \cap M$ at most in one interval (which then projects to a single point in $\hat{B}$).*

  Even without that assumption one can show that the shadow boundary $\hat{B} \subset \mathbb{R}^d$ is a $(d-1)$–regular set, but in general there may be gradient orbits contributing to several points in $\hat{B}$.

  I conjecture that the theorem holds in complete generality, without the positivity assumption.

- *For the upper bound (2) $\tilde{M}$ can be assumed to be connected, since pulling apart the components of a nonconnected M by Euclidean translations leaves the volume invariant while increasing the length of the shadow boundary.*

- *By a deformation argument one sees that for suitable constants $c_l, c_u > 0$ only depending on $\tilde{M}$ and k one can find embeddings of M with $k_{\max}(M) = k$ and arbitrarily large volume $V(M)$ with*

  $$L(M) \leq L(S_k^d) + c_l$$

  respectively

  $$L(M) \geq \frac{V(M)}{V(S_k^d)} \cdot L(S_k^d) - c_u.$$

- *The theorem can be applied in entropic estimates, see [2].*

The proof is based on the gradient flow

$$\Phi_t : M \to M \qquad (t \in \mathbb{R})$$

of the height function $h := x\restriction_M$ with vector field $X := \nabla h : M \to TM$, which is the projection

$$X = e - (N, e)N \tag{3}$$

of the vertical unit vector field $\bar{e} := \bar{\nabla} x$, $e := \bar{e}\restriction_M$ to TM.

Objects on $\mathbb{R}^{d+1}$ are marked with a bar and those on $\mathbb{R}^d$ with a hat. Whenever possible, the inner product $\bar{g}(\cdot, \cdot)$ is abbreviated by writing $(\cdot, \cdot)$.

We prepare the proof of the upper bound (2) by a sequence of lemmas.

The gradient flow changes the angle between the normal vector field N and the vertical direction e according to

$$\frac{d}{dt}(N(\Phi_t(m)), e)|_{t=0} = (A(m)X(m), e) = (A(m)X(m), X(m)). \tag{4}$$

Under our assumption $A > 0$ this derivative is strictly positive on the non-trivial orbits. Thus in that case

$$B := P^{-1}(\hat{B}) \cap M \subset \{m \in M \mid (N(m), e) = 0\}$$

is an embedded submanifold of dimension $d-1$. By strict convexity of the region bounded by M the projection P is injective on B.

We set

$$M_0 := \{m \in M \mid (N(m), e) \leq 0\}$$

and

$$M_t := \Phi_t(M_0) \qquad (t \in \mathbb{R}).$$

Then the boundary equals

$$\partial M_t = B_t := \Phi_t(B).$$

Lemma 3 $$L(M) = \int_B \mathbf{i}_X \omega.$$

Proof. $X\restriction_B = e\restriction_B$. The projection $P\restriction_B : B \to \hat{B}$ is a diffeomorphism (but not an isometry in general). Thus we have

$$\begin{aligned} \int_B \mathbf{i}_X \omega &= \int_B \mathbf{i}_e \omega = \int_B \mathbf{i}_e \mathbf{i}_N \bar{\omega}\restriction_{T_M \mathbb{R}^{d+1}} \\ &= -\int_B \mathbf{i}_N \mathbf{i}_e \bar{\omega}\restriction_{T_M \mathbb{R}^{d+1}} = -\int_B \mathbf{i}_N P^* \hat{\omega} = -\int_{\hat{B}} \mathbf{i}_{\hat{N}} \hat{\omega}, \end{aligned}$$

$\hat{N} : \hat{B} \to S^{d-1}$ being the outward normal vector field. □

Lemma 4 $$V(M) - V(M_0) \geq \int_0^\infty \left[\int_{\partial M_t} \mathbf{i}_X \omega \right] dt$$

and

$$V(M_0) \geq \int_{-\infty}^0 \left[\int_{\partial M_t} \mathbf{i}_X \omega \right] dt,$$

with equality for the sphere S_k^d.

Proof.

$$\int_M \omega \geq \lim_{t \to \infty} \int_{M_t} \omega = \int_{M_0} \omega + \int_0^\infty \left(\frac{d}{dt} \int_{M_t} \omega \right) dt$$

since every orbit of the gradient flow meets B in at most one point, and

$$\frac{d}{dt} \int_{M_t} \omega = \int_{\partial M_t} \mathbf{i}_X \omega. \tag{5}$$

The second equation follows similarly. For S_k^d every orbit except the two poles passes the equator B, hence the equality. □

Lemma 5 $\quad L_X\omega = -(N,e)\cdot \mathrm{tr}(A)\cdot\omega.$

Proof.

$$L_X\omega = (L_{\bar X}\mathbf{i}_{\bar N}\bar\omega)\restriction_{TM}$$

for an extension $\bar N$ of the unit normal vector field N to a tubular neighbourhood of M and $\bar X := \bar e - (\bar N,\bar e)\bar N$.

$$L_{\bar X}\mathbf{i}_{\bar N}\bar\omega = \mathbf{i}_{\bar N}L_{\bar X}\bar\omega + \mathbf{i}_{[\bar X,\bar N]}\bar\omega \tag{6}$$

with

$$\mathbf{i}_{\bar N}L_{\bar X}\bar\omega = \mathbf{i}_{\bar N}\mathrm{div}(\bar X)\bar\omega = \mathrm{div}(\bar X)\mathbf{i}_{\bar N}\bar\omega,$$

while

$$\mathrm{div}(\bar X) = -\mathrm{div}((\bar N,\bar e)\bar N) = -(\bar N,\bar e)\cdot\mathrm{div}(\bar N) - L_{\bar N}(\bar N,\bar e).$$

The first term is rewritten using the identity (see, e.g. Do Carmo, §6.3)

$$\mathrm{div}(\bar N)\restriction_M = d\cdot H = \mathrm{tr}(A),$$

for the mean curvature $H : M \to \mathbb{R}$. The Lie derivative of the inner product consists of three terms:

$$L_{\bar N}\bar g(\bar N,\bar e) = (L_{\bar N}\bar g)(\bar N,\bar e) + (L_{\bar N}\bar N,\bar e) + (\bar N, L_{\bar N}\bar e). \tag{7}$$

The last two terms of (7) vanish, since $L_{\bar N}\bar N = 0$ and

$$(\bar N, L_{\bar N}\bar e) = -\tfrac{1}{2}\tfrac{\partial}{\partial x}(\bar N,\bar N) = -\tfrac{1}{2}\tfrac{\partial}{\partial x}1.$$

By the general formula $(L_Y g)_{i,j} = Y_{i|j} + Y_{j|i}$ (Abraham-Marsden [1] §2.7) for the Lie derivative of a metric tensor and $(\bar N,\bar N) = 1$

$$L_{\bar N}\bar g(\bar N,\bar e) = (L_{\bar N}\bar g)(\bar N, e) = (\nabla\bar N_0,\bar N),$$

$\bar N_0$ being the vertical component of $\bar N$. So (with a slight abuse of notation) the first term in (6) equals

$$\mathbf{i}_{\bar N}L_{\bar X}\bar\omega = -\left(\mathrm{tr}(\bar A) + (\nabla\bar N_0,\bar N)\right)\mathbf{i}_{\bar N}\bar\omega. \tag{8}$$

For the second term in (6) we use the identity

$$[\bar{X}, \bar{N}] = \frac{\partial}{\partial x}\bar{N} + (\nabla \bar{N}_0, \bar{N})\bar{N}$$

and conclude from (6) and (8) that

$$L_{\bar{X}} \mathbf{i}_{\bar{N}} \bar{\omega} = -\mathrm{tr}(\bar{A}) \mathbf{i}_{\bar{N}} \bar{\omega} + \mathbf{i}_{\partial \bar{N}/\partial x} \bar{\omega}.$$

However, the d–form

$$\left(\mathbf{i}_{\partial \bar{N}/\partial x} \bar{\omega}\right) \restriction_{TM} = 0,$$

since $\partial \bar{N}/\partial x$ is tangential to M. □

Lemma 6 *For $m \in B$*

$$(N(\Phi_t(m)), e) \leq \tanh(kt) \qquad (t \geq 0), \tag{9}$$

with equality for the sphere S_k^d of radius $1/k$.

Proof. Both sides of (9) vanish for $t = 0$. Let us assume that we have equality in (9) for some t. Then by (4)

$$\begin{aligned} \frac{d}{dt}(N(\Phi_t(m)), e) &= (A(y)X(y), X(y)) \leq k \cdot (X(y), X(y)) \\ &= k \cdot \left(1 - (N(y), e)^2\right) \end{aligned} \tag{10}$$

with $y := \Phi_t(m)$, using (3). So $t \mapsto (N(\Phi_t(m)), e)$ increases fastest for the sphere S_k^d.

There the vertical component $z \in [-1/k, 1/k]$ of $y \equiv (z, y_1, \ldots, y_d) \in S_k^d$ has the form $z = k \cdot (N(y), e)$, and (10) with equality is equivalent to the differential equation for y.

The gradient equation for the sphere has the explicit solution

$$x(t) = \tanh(kt)/k, \quad x_i(t) = x_i(0)/\cosh(kt) \quad (i = 1, \ldots, d),$$

so that (9) follows. □

Proof of the Theorem. We deduce the lower bound (1) from the following observation. Denote by m a point of m on which h is maximal, so that $N(m) = e$. Then $\hat{S}$ contains a ball $\hat{K}$ of radius $1/k$ around $\hat{m} := P(m)$:

Otherwise there is a point $\hat{y} \in \mathbb{R}^d - \hat{S}$ with $\|\hat{y} - \hat{m}\| < 1/k$, and we consider the two-plane $F \subset \mathbb{R}^{d+1}$ through m containing the vertical line $L := P^{-1}(\hat{y})$. By assumption $M \cap F$ does not meet L, although the distance between m and L is $< 1/k$ and $T_m M$ is horizontal.

But this leads to a contradiction, since there is a regular curve of length $\geq \frac{1}{2}\pi/k$ in $F \cap M$ starting at m in the direction of L, which is of geodesic curvature $\leq k$.

By projecting $\hat{B}$ to $\partial\hat{K}$ along rays through $\hat{m}$, one sees that the length $L(M)$ of the shadow boundary of M is larger than that of $\partial\hat{K}$. The latter equals $L(S_k^d)$.

To show the upper bound (2), we take another time derivative of (5) and obtain, using Lemma 5

$$\frac{d^2}{dt^2}\int_{M_t} \omega = \int_{\partial M_t} \mathbf{i}_X L_X \omega = -\int_{\partial M_t} (N, e)\mathrm{tr}(A)\mathbf{i}_X \omega. \tag{11}$$

Setting $f(t) := \frac{d}{dt}\int_{M_t} \omega / L(M)$ and $f_S(t)$ the corresponding expression for the sphere S_k^d, we have by (11), Lemma 5 and Lemma 6

$$\frac{d}{dt}\ln(f_S(t)) \leq \frac{d}{dt}\ln(f(t)) < 0.$$

By Gronwall's inequality ([1], Ch. 2.1) applied to

$$f(t) = 1 + \int_0^t \frac{f'(s)}{f(s)} f(s) ds$$

we find

$$f(t) \geq f_S(t) \qquad (t \geq 0).$$

By Lemma 4

$$\frac{V(M) - V(M_0)}{L(M)} \geq \int_0^\infty f(t)dt$$

and conversely

$$\frac{V(M_0)}{L(M)} \geq \int_{-\infty}^0 f(t)dt,$$

with equality for the sphere. Together this implies

$$\frac{V(M)}{L(M)} \geq \frac{V(S_k^d)}{L(S_k^d)},$$

which proves our claim. □

References

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