The length of the shadow boundary

by

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Abstract

Among all embedded manifolds with positive exterior curvature \( \leq k \) the ratio between the length of the shadow boundary and the surface is maximized by the sphere of radius \( 1/k \).

Let \( \tilde{M} \) be a \( d \)-dimensional closed oriented manifold. We consider an embedding

\[
I : \tilde{M} \to \mathbb{R}^{d+1}, \quad M := I(\tilde{M})
\]

in Euclidean \((\mathbb{R}^{d+1}, \tilde{g})\) with the outward unit normal vector field \( N : M \to T_M \mathbb{R}^{d+1} \).

\( V(M) \) denotes the volume of \((M, g)\) w.r.t. the Riemannian metric \( g := \tilde{g}|_{T_M} \) or, equivalently, w.r.t. the \( d \)-form

\[
\omega := i_N \varpi|_{T_M \mathbb{R}^{d+1}}
\]

on \( M \). Here \( \varpi := dx_1 \wedge \ldots \wedge dx_d \) with \( \omega := dx_1 \wedge \ldots \wedge dx_d \) is the Euclidean volume on \( \mathbb{R}^{d+1} \equiv \mathbb{R} \times \mathbb{R}^d \).

The shadow of the hypersurface \( M \) under the vertical orthogonal projection \( P : \mathbb{R}^{d+1} \equiv \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) is denoted by

\[
\hat{S} := P(M) \subset \mathbb{R}^d,
\]

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and we are interested in the $d-1$-dimensional Hausdorff measure $L(M)$ of its topological boundary

$$\hat{B} := \partial \hat{S} \subset \mathbb{R}^d,$$

the length of the shadow boundary.

Fixing the volume $v = V(M)$ does not lead to an upper bound of $L(M)$:

- For $d = 1$ the manifold is a union $\hat{M} = S^1 \cup \ldots \cup S^1$ of $n$ circles, so that we have $L(M) = 2n$ if the shadow intervals are disjoint, whereas we have $V(M) = v$ for circles of radius $v/(2\pi n)$.

- For $d \geq 2$ and arbitrary $l > 0$ there is a linear transformation $f$ on $\mathbb{R}^{d+1}$ with $V(f(M)) = V(M)$ and $L(f(M)) = l$.

However, if we in addition fix the maximal exterior curvature

$$k_{\text{max}}(M) := \max_{m \in M} \|A_m\| > 0$$

of the embedding, $A_m : T_m M \to T_m M$ being the self-adjoint Peterson operator of the second fundamental form, then the length of the shadow boundary is bounded above:

**Theorem 1** For $k_{\text{max}}(M) = k$

$$L(M) \geq L(S_k^d), \quad (1)$$

and if $A$ is positive definite,

$$L(M) \leq \frac{V(M)}{V(S_k^d)} \cdot L(S_k^d), \quad (2)$$

$S_k^d$ being the $d$–sphere of radius $1/k$.

**Remarks 2**

- The assumption $A > 0$ is rather restrictive. Indeed one may weaken it by assuming that each orbit of the gradient flow (see below) meets $P^{-1}(\hat{B}) \cap M$ at most in one interval (which then projects to a single point in $\hat{B}$).

Even without that assumption one can show that the shadow boundary $\hat{B} \subset \mathbb{R}^d$ is a $(d-1)$–regular set, but in general there may be gradient orbits contributing to several points in $\hat{B}$.

I conjecture that the theorem holds in complete generality, without the positivity assumption.
For the upper bound (2) \( \hat{M} \) can be assumed to be connected, since pulling apart the components of a nonconnected \( M \) by Euclidean translations leaves the volume invariant while increasing the length of the shadow boundary.

By a deformation argument one sees that for suitable constants \( c_l, c_u > 0 \) only depending on \( \hat{M} \) and \( k \) one can find embeddings of \( M \) with \( k_{\text{max}}(M) = k \) and arbitrarily large volume \( V(M) \) with

\[
L(M) \leq L(S_k^d) + c_l
\]

respectively

\[
L(M) \geq \frac{V(M)}{V(S_k^d)} \cdot L(S_k^d) - c_u.
\]

The theorem can be applied in entropic estimates, see [2].

The proof is based on the gradient flow

\[
\Phi_t : M \to M \quad (t \in \mathbb{R})
\]

of the height function \( h := x|_M \) with vector field \( X := \nabla h : M \to TM \), which is the projection

\[
X = e - (N, e)N
\]

of the vertical unit vector field \( \varepsilon := \nabla x, e := \overline{e}|_M \) to \( TM \).

Objects on \( \mathbb{R}^{d+1} \) are marked with a bar and those on \( \mathbb{R}^d \) with a hat. Whenever possible, the inner product \( \varrho(\cdot, \cdot) \) is abbreviated by writing \((\cdot, \cdot)\).

We prepare the proof of the upper bound (2) by a sequence of lemmas.

The gradient flow changes the angle between the normal vector field \( N \) and the vertical direction \( e \) according to

\[
\frac{d}{dt}(N(\Phi_t(m)), e)|_{t=0} = (A(m)X(m), e) = (A(m)X(m), X(m)).
\]

Under our assumption \( A > 0 \) this derivative is strictly positive on the non-trivial orbits. Thus in that case

\[
B := P^{-1}(\hat{B}) \cap M \subset \{ m \in M \mid (N(m), e) = 0 \}
\]
is an embedded submanifold of dimension $d - 1$. By strict convexity of the region bounded by $M$ the projection $P$ is injective on $B$.

We set
\[ M_0 := \{ m \in M \mid (N(m), e) \leq 0 \} \]
and
\[ M_t := \Phi_t(M_0) \quad (t \in \mathbb{R}). \]
Then the boundary equals
\[ \partial M_t = B_t := \Phi_t(B). \]

**Lemma 3** \[ L(M) = \int_{\partial B} i_X \omega. \]

**Proof.** $X|_B = e|_B$. The projection $P|_B : B \to \hat{B}$ is a diffeomorphism (but not an isometry in general). Thus we have
\[ \int_{\partial B} i_X \omega = \int_{\partial B} i_e \omega = \int_{\partial B} i_e i_N \omega|_{T_M \mathbb{R}^d} = \int_{\hat{B}} i_N P^* \omega = -\int_{\hat{B}} i_N \hat{\omega}, \]
$\hat{N} : \hat{B} \to S^{d-1}$ being the outward normal vector field. \hfill \Box

**Lemma 4** \[ V(M) - V(M_0) \geq \int_0^\infty \left[ \int_{\partial M} i_X \omega \right] dt \]
and
\[ V(M_0) \geq \int_{-\infty}^0 \left[ \int_{\partial M} i_X \omega \right] dt, \]
with equality for the sphere $S^d_k$.

**Proof.**
\[ \int_M \omega \geq \lim_{t \to \infty} \int_{M_t} \omega = -\int_{M_0} \omega + \int_{M_0}^\infty \left( \frac{d}{dt} \int_{M_t} \omega \right) dt \]
since every orbit of the gradient flow meets $B$ in at most one point, and
\[ \frac{d}{dt} \int_{M_t} \omega = \int_{\partial M_t} i_X \omega. \] \hfill (5)

The second equation follows similarly. For $S^d_k$ every orbit except the two poles passes the equator $B$, hence the equality. \hfill \Box
Lemma 5 \[ L_X\omega = -(N, e) \cdot \text{tr}(A) \cdot \omega. \]

Proof.

\[ L_X\omega = (L_N i_N \bar{\omega}) |_{T_M} \]

for an extension \( \bar{N} \) of the unit normal vector field \( N \) to a tubular neighbourhood of \( M \) and \( X := \bar{e} - (\bar{N}, \bar{e}) \bar{N} \).

\[ L_X i_N \bar{\omega} = i_N L_N \bar{\omega} + i_N [N, \bar{\omega}] \] \hspace{1cm} (6)

with \( i_N L_N \bar{\omega} = i_N \text{div}(\bar{X}) \bar{\omega} = \text{div}(\bar{X}) i_N \bar{\omega} \),

while

\[ \text{div}(\bar{X}) = -\text{div}((\bar{N}, \bar{e}) \bar{N}) = -(\bar{N}, \bar{e}) \cdot \text{div}(\bar{N}) - L_N (\bar{N}, \bar{e}). \]

The first term is rewritten using the identity (see, e.g. Do Carmo, §6.3)

\[ \text{div}(\bar{N}) |_{M} = d \cdot H = \text{tr}(A), \]

for the mean curvature \( H : M \to \mathbb{R} \). The Lie derivative of the inner product consists of three terms:

\[ L_N i_N \bar{\omega} = (L_N \bar{g})(\bar{N}, \bar{e}) + (L_N, \bar{N}, \bar{e}) + (\bar{N}, L_N e). \] \hspace{1cm} (7)

The last two terms of (7) vanish, since \( L_N \bar{N} = 0 \) and

\[ (\bar{N}, L_N e) = -\frac{1}{2} \frac{\partial}{\partial \bar{e}} (\bar{N}, \bar{N}) = -\frac{1}{2} \frac{\partial}{\partial \bar{e}} 1. \]

By the general formula \( (L_Y g)_{i,j} = Y_{ij} + Y_{j|i} \) (Abraham-Marsden [1] §2.7) for the Lie derivative of a metric tensor and \( (\bar{N}, \bar{N}) = 1 \)

\[ L_N \bar{g}(\bar{N}, \bar{e}) = (L_N \bar{g})(\bar{N}, e) = (\nabla \bar{N}_0, \bar{N}), \]

\( \bar{N}_0 \) being the vertical component of \( \bar{N} \). So (with a slight abuse of notation) the first term in (6) equals

\[ i_N L_N \bar{\omega} = - (\text{tr}(A) + (\nabla \bar{N}_0, \bar{N})) i_N \bar{\omega}. \] \hspace{1cm} (8)
For the second term in (6) we use the identity
\[ [\mathcal{X}, \mathcal{N}] = \frac{\partial}{\partial x} \mathcal{N} + (\nabla \mathcal{N}_0, \mathcal{N}) \mathcal{N} \]
and conclude from (6) and (8) that
\[ L_{\mathcal{X}} i_{\mathcal{N}} \mathcal{N} = -\text{tr}(\mathcal{A})i_{\mathcal{N}} \mathcal{N} + i_{\partial \mathcal{N}/\partial x} \mathcal{N}. \]
However, the $d$–form
\[ (i_{\partial \mathcal{N}/\partial x} \mathcal{N}) \big|_{TM} = 0, \]
since $\partial \mathcal{N}/\partial x$ is tangential to $M$. □

**Lemma 6** For $m \in B$
\[ (N(\Phi_i(m)), e) \leq \tanh(kt) \quad (t \geq 0), \] (9)
with equality for the sphere $S^d_k$ of radius $1/k$.

**Proof.** Both sides of (9) vanish for $t = 0$. Let us assume that we have equality in (9) for some $t$. Then by (4)
\[
\frac{d}{dt} (N(\Phi_i(m)), e) = (A(y)X(y), X(y)) \leq k \cdot (X(y), X(y)) \\
= k \cdot (1 - (N(y), e)^2) \] (10)
with $y := \Phi_i(m)$, using (3). So $t \mapsto (N(\Phi_i(m)), e)$ increases fastest for the sphere $S^d_k$.

There the vertical component $z \in [-1/k, 1/k]$ of $y = (z, y_1, \ldots, y_d) \in S^d_k$ has the form $z = k \cdot (N(y), e)$, and (10) with equality is equivalent to the differential equation for $y$.

The gradient equation for the sphere has the explicit solution
\[ x(t) = \tanh(kt)/k, \quad x_i(t) = x_i(0)/\cosh(kt) \quad (i = 1, \ldots, d), \]
so that (9) follows. □

**Proof of the Theorem.** We deduce the lower bound (1) from the following observation. Denote by $m$ a point of $m$ on which $h$ is maximal, so that $N(m) = e$. Then $\hat{S}$ contains a ball $\hat{K}$ of radius $1/k$ around $\hat{m} := P(m)$:
Otherwise there is a point $\hat{y} \in \mathbb{R}^d - \hat{S}$ with $\|\hat{y} - \hat{m}\| < 1/k$, and we consider the two-plane $F \subset \mathbb{R}^{d+1}$ through $m$ containing the vertical line $L := P^{-1}(\hat{y})$. By assumption $M \cap F$ does not meet $L$, although the distance between $m$ and $L$ is $< 1/k$ and $T_m M$ is horizontal.

But this leads to a contradiction, since there is a regular curve of length $\geq \frac{1}{2} \pi / k$ in $F \cap M$ starting at $m$ in the direction of $L$, which is of geodesic curvature $\leq k$.

By projecting $\hat{B}$ to $\partial \hat{K}$ along rays through $\hat{m}$, one sees that the length $L(M)$ of the shadow boundary of $M$ is larger than that of $\partial \hat{K}$. The latter equals $L(S^d_k)$.

To show the upper bound (2), we take another time derivative of (5) and obtain, using Lemma 5

$$\frac{d^2}{dt^2} \int_{M_t} \omega = \int_{\partial M_t} i_X L_X \omega = - \int_{\partial M_t} (N, e) \text{tr}(A)i_X \omega. \quad (11)$$

Setting $f(t) := \frac{d}{dt} \int_{M_t} \omega / L(M)$ and $f_s(t)$ the corresponding expression for the sphere $S^d_k$, we have by (11), Lemma 5 and Lemma 6

$$\frac{d}{dt} \ln(f_s(t)) \leq \frac{d}{dt} \ln(f(t)) < 0.$$

By Gronwall’s inequality ([1], Ch. 2.1) applied to

$$f(t) = 1 + \int_0^t \frac{f'(s)}{f(s)} f(s) \, ds$$

we find

$$f(t) \geq f_s(t) \quad (t \geq 0).$$

By Lemma 4

$$\frac{V(M) - V(M_0)}{L(M)} \geq \int_0^\infty f(t) \, dt$$

and conversely

$$\frac{V(M_0)}{L(M)} \geq \int_0^\infty f(t) \, dt,$$

with equality for the sphere. Together this implies

$$\frac{V(M)}{L(M)} \geq \frac{V(S^d_k)}{L(S^d_k)},$$

which proves our claim. \qed
References

