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EXISTENCE RESULTS FOR MEAN FIELD EQUATIONS

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ABSTRACT. Let Ω be an annulus. We prove that the mean field equation

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \quad &\text{in } \Omega \\ \psi &= 0 &\text{on } \partial\Omega \end{aligned}$$

admits a solution for $\beta \in (-16\pi, -8\pi)$. This is a supercritical case for the Moser-Trudinger inequality.

1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. In this paper, we consider the following mean field equation

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \quad &\text{in } \Omega, \\ \psi &= 0, &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

for $\beta \in (-\infty, +\infty)$. (1.1) is the Euler-Lagrange equation of the following functional

$$J_\beta(\psi) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{\beta}\log\int_\Omega e^{-\beta\psi} \tag{1.2}$$

in $H_0^{1,2}(\Omega)$. This variational problem arises from Onsager's vortex model for turbulent Euler flows. In that interpretation, ψ is the stream function in the infinite vortex limit, see [MP,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if $\beta > -8\pi$. In that situation, Caglioti et al. [CLMP1] and Kiessling [K] showed the existence of a minimizer of J_β. This is based on the Moser-Trudinger inequality

$$\frac{1}{2}\int_\Omega |\nabla\psi|^2 \geq \frac{1}{8\pi}\log\int_\Omega e^{-8\pi\psi}, \qquad \text{for any } \psi \in H_0^{1,2}(\Omega), \tag{1.3}$$

which implies the relevant compactness and coercivity condition for J_β in case $\beta > -8\pi$. For $\beta \leq -8\pi$, the situation becomes different as described in [CLMP1]. On the unit disk, solutions blow up if one approaches $\beta = -8\pi$ -the critical case for (1.3)-(see also [CLMP2] and [Su]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of β for which solutions exist. On the other hand, for an annulus, [CLMP1] constructed radially symmetric solutions for any β, and the construction of Bahri-Coron [BC] makes it plausible that solutions on domains with non-trivial topology exist below -8π. Thus, for $\beta \leq -8\pi$, J_β is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case $\beta < -8\pi$ on domains with non-trivial topology.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain whose complement contains a bounded region, e.g. Ω an annulus. Then (1.1) has a solution for all $\beta \in (-16\pi, -8\pi)$.*

The solutions we find, however, are not minimizers of J_β-those do not exist in case $\beta < 8\pi$, since J_β has no lower bound-but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [CLMP1] and [K].

Certainly we can generalize Theorem 1.1 to the following equation

$$\begin{aligned} -\Delta\psi &= \frac{Ke^{-\beta\psi}}{\int_\Omega Ke^{-\beta\psi}}, \quad &&\text{in } \Omega, \\ \psi &= 0, &&\text{on } \partial\Omega, \end{aligned}$$

which was studied in [CLMP2]. Here K is a positive function on $\bar{\Omega}$.

With the same method, we may also handle the equation

$$\Delta u - c + cKe^u = 0, \qquad \text{for } 0 \leq c < \infty \tag{1.4}$$

on a compact Riemann surface Σ of genus at least 1, where K is a positive function

(1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$J_c(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 + c\int_\Sigma u - c\log\int_\Sigma Ke^u. \tag{1.5}$$

Because of the term $c\int_\Sigma u$, J_c remains invariant under adding a constant to u, and therefore we may normalize u by the condition

$$\int_\Sigma Ke^u = 1$$

which explains the absence of the factor $(\int Ke^u)^{-1}$ in (1.4). $c < 8\pi$ again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case $c = 8\pi$ yields the so-called Kazdan-Warner equation [KW] and was treated in [DJLW] and [NT] by giving sufficient conditions for the existence of a minimizer of $J_{8\pi}$. Here, we construct again saddle point type critical points to show

Theorem 1.2. *Let Σ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for $8\pi < c < 16\pi$.*

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value α_β, which is bounded below by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [St1] and [St2], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a coercivity condition and show that α_β is achieved by some u_β for $\beta \in \Lambda$. Next, for any fixed $\bar\beta \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \subset \Lambda$ tending to $\bar\beta$, with the help of results in [BM] and [LS] we show that u_{β_k} subconverges strongly to some $u_{\bar\beta}$ which achieves $\alpha_{\bar\beta}$.

After completing our paper, we were informed that Struwe and Tarantello [ST] obtained a non-constant solution of (1.4), when Σ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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2. Minimax values

Let $\rho = -\beta$ and $u = -\beta\psi$. We rewrite (1.1) as

$$\begin{aligned} -\Delta u &= \rho \frac{e^u}{\int_\Omega e^u}, \quad && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

and (1.2) as

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \rho \log \int_\Omega e^u \tag{2.2}$$

for $u \in H_0^{1,2}(\Omega)$.

It is easy to see that J_ρ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (8\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of u by

$$m_c(u) = \frac{\int_\Omega x e^u}{\int_\Omega e^u}.$$

Let B be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that B is the unit disk centered at the origin. Then we define a family of functions

$$h : D \to H_0^{1,2}(\Omega)$$

satisfying

$$\lim_{r\to 1} J_\rho(h(r,\theta)) \to -\infty \tag{2.3}$$

and

$$\lim_{r\to 1} m_c(h(r,\theta)) \text{ is a continuous curve enclosing } B. \tag{2.4}$$

Here $D = \{(r,\theta)|0 \le r < 1, \theta \in [0,2\pi)\}$ is the open unit disk. We denote the set of all such families by $\mathcal{D}_\rho$. It is easy to check that $\mathcal{D}_\rho \neq \emptyset$. Now we can define a minimax value

$$\alpha_\rho := \inf_{h\in\mathcal{D}_\rho} \sup_{u\in h(D)} J_\rho(u).$$

The following lemma will make crucial use of the non-trivial topology of Ω, more precisely of the fact that the complement of Ω has a bounded component.

Lemma 2.1. *$\alpha_\rho > -\infty$ for any $\rho \in (8\pi, 16\pi)$.*

Remark. It is an interesting question weather $\alpha_{16\pi} = -\infty$.

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [CL] (see also [A]). Here we have to modify a little bit.

Lemma 2.2. *Let S_1 and S_2 be two subsets of $\bar\Omega$ satisfying $dist(S_1,S_2) \ge \delta_0 > 0$ and $\gamma_0 \in (0,1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon,\delta_0,\gamma_0) > 0$ such that*

$$\int_\Omega e^u \le c\exp\{\frac{1}{32\pi-\epsilon}\int_\Omega |\nabla u|^2 + c\}$$

holds for all $u \in H_0^{1,2}(\Omega)$ satisfying

$$\frac{\int_{S_1} e^u}{\int_\Omega e^u} \ge \gamma_0 \qquad \text{and} \qquad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \ge \gamma_0. \tag{2.5}$$

Proof. The Lemma follows from the argument in [CL] and the following Moser-Trudinger inequality

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \ge c \tag{$*$}$$

for any $u \in H_0^{1,2}(\Omega)$, where c is a constant independent of $u \in H_0^{1,2}(\Omega)$. □

We will discuss the inequality ($*$) and its application in another paper.

Proof of Lemma 2.1. For fixed $\rho \in (8\pi, 16\pi)$ we claim that there exists a constant c_ρ such that

$$\sup_{u\in h(D)} J_\rho(u) \ge c_\rho, \qquad \text{for any } h \in \mathcal{D}_\rho. \tag{2.6}$$

Clearly (2.6) implies the Lemma. By the definition of h, for any $h \in \mathcal{D}_\rho$, there exists $u \in h(D)$ such that

$$m_c(u) = 0.$$

We choose $\epsilon > 0$ so small that $\rho < 16\pi - 2\epsilon$. Assume (2.6) does not hold. Then we have sequences $\{h_i\} \subset \mathcal{D}_\rho$ and $\{u_i\} \subset H_0^{1,2}(\Omega)$ such that $u_i \in h_i(D)$ and

$$m_c(u_i) = 0 \tag{2.7}$$

$$\lim_{i\to\infty} J(u_i) = -\infty. \tag{2.8}$$

We have the following Lemma.

Lemma 2.3. *There exists $x_0 \in \bar\Omega$ such that*

$$\lim_{i\to\infty} \frac{\int_{B_{1/2}(x_0)\cap\Omega} e^{u_i}}{\int_\Omega e^{u_i}} \to 1. \tag{2.9}$$

Proof. Set

$$A(x) := \lim_{i\to\infty} \frac{\int_{B_{1/4}(x)\cap\Omega} e^{u_i}}{\int_\Omega e^{u_i}}.$$

Assume that the Lemma were false, then there exists $x_0 \in \bar\Omega$ such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \Omega.$$

It is easy to check $A(x_0) > 0$, since Ω can be covered by finite many balls of radius 1/4. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

$$\frac{\int_{\Omega\setminus B_{1/2}(x_0)} e^{u_i}}{\int_\Omega e^{u_i}} \to 0 \tag{2.10}$$

as $i \to \infty$, which implies (2.9). $\square$

Now we continue to prove Lemma 2.1. (2.9) implies

$$\begin{aligned}\frac{\int_\Omega x e^{u_i}}{\int_\Omega e^{u_i}} - x_0 &= \frac{\int_\Omega (x-x_0)e^{u_i}}{\int_\Omega e^{u_i}} \\ &= \frac{\int_{B_{1/2}(x_0}(x-x_0)e^{u_i}}{\int_\Omega e^{u_i}} + o(1)\end{aligned}$$

which, in turn, implies that $|m_c(u_i) - x_0| < 2/3$. This contradicts (2.7). $\square$

Lemma 2.4. *α_ρ/ρ is non-increasing in $(8\pi, 16\pi)$.*

Proof. We first observe that if $J(u) \leq 0$, then $\log \int_\Omega e^u > 0$ which implies that

$$J_\rho(u) \geq J_{\rho'}(u) \qquad \text{for } \rho' \geq \rho.$$

Hence $\mathcal{D}_\rho \subset \mathcal{D}_{\rho'}$ for any $16\pi > \rho' \geq \rho > 8\pi$. On the other hand, it is clear that

$$\frac{J_\rho}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2}(\frac{1}{\rho} - \frac{1}{\rho'}) \int_\Omega |\nabla u|^2 \geq 0,$$

if $\rho' \geq \rho$. Hence we have

$$\frac{\alpha_\rho}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}$$

for $16\pi > \rho' \geq \rho > 8\pi$. $\square$

3. Existence for a dense set

In this section we show that α_ρ is achieved if ρ belongs to a certain dense subset of $(8\pi, 16\pi)$ defined below.

The crucial problem for our functional is the lack of a coercivity condition, *i.e.* for a Palais-Smale sequence u_i for J_ρ, we do not know whether $\int_\Omega |\nabla u_i|^2$ is bounded.

We first have the following lemma.

Lemma 3.1. *Let u_i be a Palais-Smale sequence for J_ρ, i.e. u_i satisfies*

$$|J_\rho(u_i)| \le c < \infty \tag{3.1}$$

and

$$dJ_\rho(u_i) \to 0 \text{ strongly in } H^{-1,2}(\Omega) \tag{3.2.}$$

If, in addition, we have

$$\int_\Omega |\nabla u_i|^2 \le c_0, \qquad \text{for } i = 1, 2, \cdots \tag{3.3}$$

for a constant c_0 independent of i, then u_i subconverges to a critical point u_0 for J_ρ strongly in $H_0^{1,2}(\Omega)$.

Proof. The proof is standard, but we provide it here for convenience of the reader.

Since $\int_\Omega |\nabla u_i|^2$ is bounded, there exists $u_0 \in H_0^{1,2}(\Omega)$ such that

(i) u_i converges to u_0 weakly in $H_0^{1,2}(\Omega)$,
(ii) u_i converges to u_0 strongly in $L^p(\Omega)$ for any $p > 1$ and almost everywhere,
(iii) e^{u_i} converges to e^{u_0} strongly in $L^p(\Omega)$ for any $p \ge 1$.

¿From (i)-(iii), we can show that $dJ(u_0) = 0$, *i.e.* u_0 satisfies

$$-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.$$

Testing dJ_ρ with $u_i - u_0$, we obtain

$$\begin{aligned} o(1) &= \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \Big(\frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}}\Big)(u_i - u_0) \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1), \end{aligned}$$

by (i)-(iii). Hence u_i converges to u_0 strongly in $H_0^{1,2}(\Omega)$. □

Since by Lemma 2.4 $\rho \to \alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$, $\rho \to \alpha_\rho/\rho$ is a.e. differentiable. Set

$$\Lambda := \{\rho \in (8\pi, 16\pi) | \alpha_\rho/\rho \text{ is differentiable at } \rho\}. \tag{3.4}$$

$\bar\Lambda = [8\pi, 16\pi]$, see [St1]. Let $\rho \in \Lambda$ and choose $\rho_k \nearrow \rho$ such that

$$0 \le \lim_{k\to\infty} -\frac{1}{(\rho - \rho_k)}\Big(\frac{\alpha_\rho}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k}\Big) \le c_1 \tag{3.5}$$

for some constant c_1 independent of k.

Lemma 3.2. *α_ρ is achieved by a critical point u_ρ for J_ρ provided that $\rho \in \Lambda$.*

Proof. Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists $\delta > 0$ such that

$$\|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta \tag{3.6}$$

in

$$N_\delta := \{u \in H_0^{1,2}(\Omega) | \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta\}.$$

Here, c_2 is any fixed constant such that $N_\delta \neq \emptyset$. Let $X_\rho : N_\delta \to H_0^{1,2}(\Omega)$ be a pseudo-gradient vector field for J_ρ in N_δ, *i.e.* a locally Lipschitz vector field of norm $\|X_\rho\|_{H_0^{1,2}} \leq 1$ with

$$\langle dJ_\rho(u), X_\rho(u)\rangle < -\delta. \tag{3.7}$$

See [P] for the construction of X_ρ.

Since

$$\begin{aligned}\|dJ_\rho(u) - dJ_{\rho_k}(u)\| &= \|dJ_\rho - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + \|(1-\frac{\rho}{\rho_k})dJ_{\rho_k}(u)\| \\ &\leq \frac{1}{2}(1-\frac{\rho}{\rho_k})\int |\nabla u|^2 + c(1-\frac{\rho}{\rho_k})\int_\Omega |\nabla u|^2 \to 0\end{aligned}$$

uniformly in $\{u | \int_\Omega |\nabla u|^2 \leq c_2\}$, X_ρ is also a pseudo-gradient vector field for J_{ρ_k} in N_δ with

$$\langle dJ_{\rho_k}(u), X_\rho(u)\rangle < -\delta/2, \tag{3.8}$$

for $u \in N_\delta$, provided that k is sufficiently large.

For any sequence $\{h_k\}$, $h_k \in \mathcal{D}_{\rho_k} \subset \mathcal{D}_\rho$ such that

$$\sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k \tag{3.9}$$

and all $u \in h_k(D)$ such that

$$J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k), \tag{3.10}$$

we have the following estimate

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u|^2 &= \rho \cdot \rho_k \frac{\frac{J_{\rho_k}(u)}{\rho_k} - \frac{J_\rho(u)}{\rho}}{\rho - \rho_k} \\ &\leq \rho \cdot \rho_k \frac{\frac{\alpha_{\rho_k}}{\rho_k} - \frac{\alpha_\rho}{\rho}}{\rho - \rho_k} + (\rho + \rho_k) \\ &\leq C\end{aligned} \tag{3.11}$$

by (3.5), (3.9) and (3.10), where $C = (16\pi)^2 c_1 + 32\pi$.

Now we consider in N_δ the following pseudo-gradient flow for J_ρ. First choose a Lipschitz continuous cut-off function η such that $0 \le \eta \le 1$, $\eta = 0$ outside N_δ, $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H_0^{1,2}(\Omega)$ generated by ηX_ρ

$$\begin{aligned} \frac{\partial \phi}{\partial t}(u,t) &= \eta(\phi(u,t)) X_\rho(\phi(u,t)) \\ \phi(u,0) &= u. \end{aligned}$$

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$\frac{d}{dt} J_\rho(\phi(u,t))_{|t=0} \le -\delta \tag{3.12}$$

and

$$\frac{d}{dt} J_{\rho_k}(\phi(u,t))_{|t=0} \le -\delta/2 \tag{3.13}$$

for large k.

It is clear that for any $h \in \mathcal{D}_{\rho_k}$ $h(r,\theta) \notin N_\delta$ for r close to 1. Hence $\phi(h,t) \in \mathcal{D}_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot,t)$ preserves the class of $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in \mathcal{D}_\rho$ by definition

$$\sup_{u \in h(D)} J_\rho(u) \ge \alpha_\rho.$$

Hence for any $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9), $\sup_{u \in \phi(h(D),t)} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that k is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt} \sup\{J_\rho(u) | u \in \phi(h(D),t)\} \le -\delta$$

for all $t \ge 0$, which is a contradiction. □

4. Proof of Theorem 1.1

¿From section 3, we know that for any $\bar\rho \in (8\pi, 16\pi)$ there exists a sequence $\rho_k \nearrow \bar\rho$ such that α_{ρ_k} is achieved by u_k. Consequently u_k satisfies

$$\begin{aligned} -\Delta u_k &= \rho_k \frac{e^{u_k}}{\int_\Omega e^{u_i}}, \quad \text{in } \Omega, \\ u_k &= 0, \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

¿From Lemma 2.4, we have

$$J_{\bar\rho}(u_k) = \alpha_{\rho_k} \text{ is bounded.} \tag{4.2}$$

for some constant $c_0 > 0$ which is independent of k. Let $v_k = u_k - \log \int_\Omega e^{u_k}$. Then v_k satisfies

$$-\Delta v_k = \rho_k e^{v_k} \tag{4.3}$$

with

$$\int_\Omega e^{v_k} = 1. \tag{4.4}$$

By results of Brezis-Merle [BM] and Li-Shafir [LS] we have

Lemma 4.1. *([BM], [LS]) There exists a subsequence (also denoted by v_k) satisfying one of the following alternatives:*

(i) *$\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$v_k \to -\infty$ uniformly on any compact subset of Ω;*
(iii) *there exists a finite blow-up set $\Sigma = \{a_1, \cdots, a_m\} \subset \Omega$ such that, for any $1 \le i \le m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\Omega \setminus \Sigma$. Moreover,*

$$\rho_k \int_\Omega e^{v_k} \to \sum_{i=1}^m 8\pi n_i \tag{4.5}$$

where n_i is positive integer.

For our special functions v_k, we can improve Lemma 4.1 as follows

Lemma 4.2. *There exists a subsequence (also denoted by v_k) satisfying one of the following alternatives:*

(i) *$\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$v_k \to -\infty$ uniformly on $\bar{\Omega}$;*
(iii) *there exists a finite blow-up set $\Sigma = \{a_1, \cdots, a_m\} \subset \bar{\Omega}$ such that, for any $1 \le i \le m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\bar{\Omega} \setminus \Sigma$. Moreover, (4.5) holds.*

Proof. From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of Ω. There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of $\Delta u = e^u$ in a half plane; Another is bubbling slow such that after rescaling we obtain a solution of $\Delta u = e^u$ in $\mathbb{R}^2$. One can exclude the first case. In the second case, one can follow the idea in [LS] to show that (4.5) holds. See also [L]. □

Proof of Theorem 1.1. (4.4), (4.5) and $\bar{\rho} \in (8\pi, 16\pi)$ imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$. Now we can again apply Lemma 2.2 as follows.

Let S_1 and S_2 be two disjoint compact subdomains of Ω. Since $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$, we have

$$\frac{\int_{S_i} e^{u_k}}{\int_\Omega e^{u_k}} = \int_{S_i} e^{v_k} \ge c_0, \qquad i = 1, 2$$

for a constant $c_0 = c_0(S_1, S_2, \Omega) > 0$ independent of k. Choosing ϵ such that $16\pi - \bar{\rho} > 2\epsilon$ and applying Lemma 2.2, with the help of (4.2), we obtain

$$\begin{aligned} c \ge J_{\rho_k}(u_k) &= \frac{1}{2}\int_\Omega |\nabla u_k|^2 - \rho_k \log \int_\Omega e^{u_k} \\ &\ge \frac{1}{2}\left(1 - \frac{\rho_k}{16\pi - \epsilon/2}\right)\int_\Omega |\nabla u|^2 \\ &\ge \frac{1}{2}\left(1 - \frac{\bar{\rho}}{16\pi - \epsilon/2}\right)\int_\Omega |\nabla u|^2 \end{aligned}$$

which implies that $\int_\Omega |\nabla u_k|^2$ is bounded. Now by the same argument in the proof of Lemma 3.1, u_k subconverges to $u_{\bar\rho}$ strongly in $H_0^{1,2}(\Omega)$ and $u_{\bar\rho}$ is a critical point of $J_{\bar\rho}$. Clearly, $u_{\bar\rho}$ achieves $\alpha_{\bar\rho}$. This finishes the proof of Theorem 1.1.
□

Proof of Theorem 1.2. Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let Σ be a Riemann surface of positive genus. We embed $X : \Sigma \to \mathbb{R}^N$ for some $N \geq 3$ and define the center of mass for a function $u \in H^{1,2}(\Sigma)$ by

$$m_c(u) = \frac{\int_\Sigma X e^u}{\int_\Sigma e^u}.$$

Since Σ is of positive genus, we can choose a Jordan curve Γ^1 on Σ and a closed curve Γ^2 in $\mathbb{R}^N \setminus \Sigma$ such that Γ^1 links Γ^2. We know that $\inf_{u\in H^{1,2}(\Sigma)} J_c(u)$ is finite if and only if $c \in [0, 8\pi]$ (see [DJLW]). Now define a family of functions $h : D \to H^{1,2}(\Sigma)$ (as in section 2) satisfying

$$\lim_{r\to 1} J_\rho(h(r,\theta)) \to -\infty$$

and

$$\lim_{r\to 1} m_c(h(r,\theta)) \text{ as a map from } S^1 \to \Gamma^1 \text{ is of degree 1.}$$

Let $\mathcal{D}_c$ denote the set of all such families. It is also easy to check that $\mathcal{D}_c \neq \emptyset$. Set

$$\alpha_c := \inf_{h\in\mathcal{D}_c} \sup_{u\in h(D)} J_c(u).$$

We first have

$$\alpha_c > -\infty,$$

using the fact that Γ^1 links Γ^2 and Lemma 2.2. Then by the same method as presented above, we can prove that α_c is achieved by some $u_c \in H^{1,2}(\Sigma)$, which is a solution of (1.4), for $c \in (8\pi, 16\pi)$. □

References

[A] Aubin, T., *Nonlinear analysis on manifolds*, Springer-Verlag, 1982.

[BC] Bahri, A. and Coron, J. M., *Sur une equation elliptique non lineaire avec l'exposant critique de Sobolev*, C. R. Acad. Sci. Paris Ser. I **301** (1985), 345-348.

[BM] Brezis, H. and Merle, F., *Uniform estimates and blow up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Diff. Equat. **16** (1991), 1223-1253.

[CLMP1] Caglioti, E., P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Commun. Math. Phys. **143** (1992), 501-525.

[CLMP2] Caglioti, E., P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*, Commun. Math. Phys. **174** (1995), 229-260.

[CL] Chen, W. X. and Li, C., *Prescribing Gaussian curvature on surfaces with conical singularities*, J. Geom. Anal. **1** (1991), 359-372.

[DJLW] Ding, W., Jost, J., Li, J. and Wang, G., *The differential equation* $\Delta u = 8\pi - 8\pi h e^u$ *on a compact Riemann surface*, Asian J. Math. **1** (1997), 230-248.

[KW] Kazdan, J. and Warner, F., *Curvature functions for compact 2-manifolds*, Ann. Math. **99** (1974), 14–47.

[K] Kiessling, M. K. H., *Statistical mechanics of classical particles with logarithmic interactions*, Comm. Pure Appl. Math. **46** (1993), 27-56.

[L] Li, YanYan, $-\Delta u = \lambda(\frac{Ve^u}{\int_M Ve^u} - W)$ *on Riemann surfaces, preprint.*

[LS] Li, YanYan and Shafrir, I., *Blow-up analysis for solutions of* $-\Delta u = Ve^u$ *in dimension two*, Indiana Univ. Math. J. **43** (1994), 1255-1270.

[MP] Marchioro, C. and M. Pulvirenti,, *Mathematical theory of incompressible nonviscous fluids*, Appl. Math. Sci. **96**, Springer-Verlag, 1994.

[M] Moser, J., *A sharp form of an inequality of N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077-1092.

[NT] Nolasco, M. and Tarantello, G., *On a sharp Sobolev type inequality on two dimensional compact manifolds, preprint.*

[P] Palais R. S., *Critical point theory and the minimax principle*, Global Analysis (Proc. Sympos. Pure Math.), vol. 15, 1968, pp. 185-212.

[St1] Struwe, M., *The evolution of harmonic mappings with free boundaries*, Manuscr. Math. **70** (1991), 373-384.

[St2] Struwe, M., *Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature*, Analysis, et cetera (Rabinowitz, P. H. and Zehnder, E., eds.), 1990, pp. 639-666.

[ST] Struwe, M. and G. Tarantello, *On multivortex solutions in Chern-Simons gauge theory, preprint.*

[Su] Suzuki, T., *Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity*, Ann. Inst. H. Poincaré, Anal. Non Lineaire **9** (1992), 367-398.

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